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Elliptic Weighted Problem with Indefinite Asymptotically Linear Nonlinearity

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Abstract: The objective of this paper is the study the following nonlinear elliptic problem involving a weight function:

$-div(a(x)\nabla \upsilon) = f(x, u)$ *in* Ω *and* $u = 0$ *on* $\partial\Omega$ $\qquad (P)$

where, Ω is a regular bounded subset $\mathbb{R}^N$ and $N \geq 2$, $a(x)$ is a nonnegative function and $f(x, t)$ is allowed to be sign-changing. We employ variational techniques to prove the existence of a nontrivial solution for the problem (P), under some suitable assumptions, when the nonlinearity is asymptotically linear. Then, we prove by the same method the existence of positive solution when the function f is superlinear and subcritical at infinity.

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Introduction with Main Results

In the present paper, we investigate the existence result for the following nonlinear elliptic equation involving sign-changing nonlinearity:

$$\begin{cases} -div\left(a(x)\nabla u\right) = f(x,u) & in\ \ \Omega, \\ \qquad\qquad u = 0 & on\ \ \partial\Omega, \end{cases} \tag{1.1}$$

where, $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a smooth bounded open set, $a(x)$ is a nonnegative function and also the function $f(x, t)$ is an indefinite nonlinearity.

This problem is nonlinear and in Physics, Dynamics and Biology nonlinear problems have many interest since they are able to explain the evolution of a system. If we change some parameters or the nonlinearity, the system undergoes transitions mainly the existence of solutions: We have the bifurcation phenomena. The problem (1.1) is the stationary position of the problem induced in 1952 by (Turing, 1952):

$$\frac{\partial u}{\partial t} - div\left(a(x)\nabla u\right) = f(x,u)$$

which modules the interaction between species and chemicals in a morphogenesis phenomenon in Biology, where u is the density and $f(x, t)$ represents the diffusion-interaction of substances. The term $-div(a(x)\nabla u)$ indicatess the substance of diffusion through the given system.

When a is a constant function, many authors have studied the problem (1.1) with asymptotically linear nonlinearity. For this reason, (Mironescu and Radulescu, 1993; 1996) considered the problem:

$$\begin{cases} -\Delta u = \lambda g(u) & in\ \ \Omega \\ \quad u = 0 & on\ \ \partial\Omega, \end{cases} \tag{1.2}$$

and supposed the assumptions:

(G1) g: $\mathbb{R} \to \mathbb{R}$ a C^1 positive function
(G2) The function g is increasing
(G3) The function g is convex
(G4) $\lim\limits_{t\to+\infty} \frac{g(t)}{t} = \ell \in (0, +\infty)$.

When f is superlinear and $\ell = (0, +\infty)$, the problem (1.2) was studied in (Brezis *et al*., 1996; Martel, 1997) and the references therein and it is generated to the p-Laplace operator in (Filippakis and Papageorgiou, 2006; Sanchón, 2007). The same problem with Bi-Laplace operator has been studied in (Arioli *et al*., 2005; Abid *et al*., 2008; Saanouni and Trabelsi, 2016b; Wei, 1996).

When the function $a(x)$ is a smooth on $\bar{\Omega}$ and $f(x, t) = \lambda g(t)$, with the same conditions (G1)-(G4), the problem

(1.1) was studied by (Saanouni and Trabelsi, 2016a). The condition $g(0) > 0$ was capital in their work.

On the other hand, the problem (1.1) was treated by (Zhou, 2002), when a is a constant function but the asymptotically linear nonlinearity depends on x and t. More precisely, the author consider the case when:

(F1) The function $f(x, t)$ is continuous on $\bar{\Omega}\times\mathbb{R}$, $f(x, t) \equiv 0$ for $t \leq 0$ and $f(x, t) \geq 0$ for all $t > 0$, for all $x \in \bar{\Omega}$

(F2) $\lim_{t\to 0}\frac{f(x,t)}{t} = p(x)$, with $0 \leq p(x) \in L^\infty(\Omega)$ and $\|p(x)\|_\infty < \lambda_1$, with $\lambda_1 > 0$ is the first eigenvalue of $\left(-\Delta, H_0^1(\Omega)\right)$

(F3) $\lim_{t\to +\infty}\frac{f(x,t)}{t} = \ell < \infty$ uniformly in a.e. $x\in\Omega$;

(F4) $\frac{f(x,t)}{t}$ is nondecreasing with respect to t in $(0, +\infty)$, for a.e. $x\in\Omega$.

and he proved also that the bifurcation phenomena occurs.

As a recent work, we can cite that (Li and Huang, 2019) a generalized quasilinear Schrödinger equations with asymptotically linear nonlinearities. They supposed that the nonlinearities $f(t)$ depend only on t and they proved that the problem has positive solutions. For the superlinear nonlinearities, the Schrödinger equations was investigated by (Li *et al*., 2020). Throughout this paper, we assume different type of conditions. The nonlinearity $f(x, t)$ does not have to be positive and it is asymptotically linear (ℓ finite) or super linear at $\pm\infty$. More precisely, we make the following assumptions:

(V1) $f \in C\left(\bar{\Omega}\times\mathbb{R}\right)$; $f(x, 0) = 0$ and $f(x, t)t \geq 0$ for all $(x, t)\in \Omega\times\mathbb{R}$;

(V2) $\lim_{t\to 0}\frac{f(x,t)}{t} < \lambda_1$, uniformly for $x\in\Omega$, where λ_1 is the first eigenvalue of the operator $-div(a(x)\nabla)$, with Dirichlet boundary condition on $\partial\Omega$

(V3) $\lim_{t\to 0}\frac{f(x,t)}{t} = \ell$, uniformly for $x\in\Omega$.

(V4) $\lim_{t\to +\infty}\frac{f(x,t)}{t^r - 1} = 0$, uniformly in x for some $r\in(2, 2^*)$, here and hereafter:

$$2^* = \begin{cases} \frac{2N}{N-2} & if\ N > 2 \\ +\infty & if\ N = 2. \end{cases}$$

Also, the weight function $a(x)$ is nonnegative and for this reason we will use weighted Sobolev spaces.

We remark that a non-trivial solution for the Eq. (1.1) is nonzero a critical point of the following functional:

$$J(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2\, dx - \int_\Omega F(x,u)dx,$$

with:

$$F(x,t) = \int_0^t f(x,s)ds.$$

In order to investigate the existence of nonzero critical point of J, we will apply Mountain Pass Theorem introduced by (Ambrosetti and Rabinowitz, 1973). The most difficult property that J has to satisfy the compactness condition, which is also called the Palais-Smale condition and often, one requires a technical condition introduced as this introduced in (Ambrosetti and Rabinowitz, 1973; Rabinowitz, 1986) and called Ambrosetti - Rabionowitz condition, that is:

$$0 < \theta F(x,t) \leq f(x,t)t,\ for\ all\ |t| \geq t_0\ and\ x\in\Omega, \qquad (AR)$$

for some $\theta > 2$ and $t_0 > 0$.

Sometimes, other type of condition were made as in the following papers (Costa and Magalhaes, 1994; Costa and Miyagaki, 1995; Jeanjean, 1999; Schechter, 1995; Stuart and Zhou, 1996; 1999). When the nonlinearity is asymptotically linear, we can not suppose the condition (AR) because it gives:

$$\lim_{t\to+\infty}\frac{F(x,t)}{t^2} = +\infty$$

and so:

$$\lim_{t\to+\infty}\frac{f(x,t)}{t} = +\infty.$$

In this study, we will not use (AR) or any assumption when we prove the existence of critical point for the functional J.

Our results state as follows.

Theorem 1.1

Assume that (V 1), (V 2) and (V 3) are satisfied and $\ell\in(0, 1)$. Then, we have:

(i) If $0 < \ell < \lambda_1$ and the following condition holds
(V 5) $\frac{f(x,t)}{t}$ is nondecreasing function with respect to t in $(0, +\infty)$ and nonincreasing in $(-\infty, 0)$.
Then, there is no solution with one sign for problem (1.1)

(ii) If $\ell > \lambda_1$, then the problem (1.1) admit a non-trivial solution

(iii) If $\ell = \lambda_1$ and (V 5) holds, then problem (1.1) admit a positive solution $u \in H_0^1(\Omega, a)$ (resp. negative solution) if and only if there exists a constant $c_0 > 0$ (resp. $c_0 < 0$) such that $u = c_0\varphi_1$ and $f(x, u) = \lambda_1 u$, with φ_1 is a positive eigenfunction associated to λ_1 (see section 2).

Theorem 1.2

Assume that (V 1) - (V 5) are satisfied and $\ell = +\infty$. Then the problem (1.1) admit a positive solution.

In the present paper C and C_i denotes positive constants, which may change from line to another.

Variational Formulation

Consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a regular bounded open set and throughout this paper, we denote:

$$\|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}}, \ for\ 1 \leq p < \infty \ and\ u \in L^p(\Omega).$$

Let $a(x) \in L^1(\Omega)$ be a nonnegative function and followed by (Calanchi *et al*., 2017), set:

$$H_0^1(\Omega, a) = cl\left\{u \in C_0^\infty(\Omega); \int_\Omega a(x)|\nabla u|^2 < \infty.\right\} \tag{2.1}$$

Set:

$$< u, \upsilon >= \int_\Omega a(x)\nabla u.\nabla \upsilon\, dx$$

and the norm:

$$\|u\| = \left(\int_\Omega a(x)|\nabla u|^2 dx\right)^{\frac{1}{2}}.$$

Weighted Sobolev spaces have been developed and studied for a long time and we can refer to (Drabek *et al*., 1997; Kufner, 1985).

For completeness, recall that the space $H_0^1(\Omega, a)$ is a Hilbert space and the embedding $H_0^1(\Omega, a) \hookrightarrow H_0^1(\Omega)$ is continuous and so there exists a constant C such that $\|u\|_{H_0^1} \leq C\|u\|$ for all $u \in H_0^1(\Omega, a)$, where $\|u\|_{H_0^1}$ the standard norm on $H_0^1(\Omega)$. Also, for $q \in [2, 2^*]$ the embedding:

$$H_0^1(\Omega, a) \hookrightarrow L^q(\Omega)$$

is continuous but it is compact if $q \in [2, 2^*)$.

Let J: $H_0^1(\Omega, a) \to \mathbb{R}$ be functional of class C^1 defined by:

$$J(u) = \frac{1}{2}\int_\Omega a(x)|\nabla u|^2 dx - \int_\Omega F(x, u)dx, \tag{2.2}$$

where:

$$F(x, s) = \int_0^s f(x, t)dt.$$

Definition 2.1

$u \in H_0^1(\Omega, a)$ is called solution of the Eq. (1.1) if for all $\varphi \in H_0^1(\Omega, a)$:

$$\int_\Omega a(x)\nabla u \nabla \varphi dx = \int_\Omega f(x, u)\varphi dx, \tag{2.3}$$

Hence, a solution of the problem (1.1) can be found as critical point of functional J. Before starting the Mountain Pass Theorem, we introduce the following definition.

Definition 2.2

Let H be a Banach space and a functional $J \in C^1(H, \mathbb{R})$. We say J satisfies the Palais Smale (PS) condition if any sequence $\{u_n\} \subset H$ such that $J(u_n)$ converges in $\mathbb{R}$ and $J'(u_n) \to 0$ in H', the dual space of H, the sequence $\{u_n\}$ has a convergent subsequence.

Proposition 2.1 (Mountain Pass Theorem (Ambrosetti and Rabinowitz, 1973))

Let H be a Banach space, J a functional in $C^1(H, \mathbb{R})$ satisfies the (PS) condition, $J(0) = 0$ and:

(i) There exist ρ, $\alpha > 0$ such that $J(u) \geq \alpha$, for all u in the boundary of $B(0, \rho)$
(ii) There exists $y \in H \backslash B(0, \rho)$ such that $J(y) < 0$

Then, the functional J admit a critical point $x \in H$ such that $J(x) \geq \alpha > 0$.

At the end of this section, recall that φ_1 denotes a normalised positive eigenfunction associated to the first eigenvalue λ_1. φ_1 satisfies:

$$\begin{cases} -div\left(a(x)\nabla\varphi_1\right) = \lambda_1\varphi_1 \ \ in\ \Omega \\ \qquad \varphi_1 = 0 \quad on\ \partial\Omega \\ \qquad \|\varphi_1\|_2 = 1. \end{cases} \tag{2.4}$$

Proof of the Theorem 1.1

We begin by proving the two geometric properties.

Lemma 3.1

Assume that (V 1) - (V 3) hold and $\ell \in (0, +\infty)$. Then, we can find two positive numbers $\alpha > 0$ and $\rho > 0$ satisfying:

$$J(u) > \alpha, \ \ forall\ u \in H_0^1(\Omega, a); \|u\| = \rho.$$

Proof

Using (V 2), there exist $\varepsilon_0 \in (0, 1)$ and δ_0 satisfying:

$$F(x,t) \le \frac{1}{2}\lambda_1 (1-\varepsilon_0)t^2 \tag{3.1}$$

for all $|t| \le \delta_0$.

From (V 3), for $1 \le q \le \frac{N+2}{N-2}$, we can have a constant $C > 0$ verifying:

$$F(x,t) \le C|t|^{q+1} \tag{3.2}$$

for all $|t| \ge \delta_0$.

So, we have:

$$F(x,t) \le \frac{1}{2}\lambda_1 (1-\varepsilon_0)t^2 + C|t|^{q+1}, \textit{ for all } t \textit{ for all } \mathbb{R}$$

and then:

$$J(u) \ge \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_1 (1-\varepsilon_0)\|u\|_2^2 - C\|u\|_{q+1}^{q+1}$$

Since $\lambda_1 \|u\|_2^2 \le \|u\|^2$ (by definition of λ_1) and by continuous embedding result, we get:

$$J(u) \ge \frac{1}{2}\varepsilon_0 \|u\|^2 - C_1 \|u\|^{q+1}. \tag{3.3}$$

Set $\|u\| = \rho > 0$ small enough, we get $J(u) \ge \alpha$, where $\alpha = \frac{1}{2}\varepsilon_0\rho^2 - C_1\rho^{q+1}$.

In the next lemma, we prove the second geometry property.

Lemma 3.2

Suppose that (V 1) and (V 3) hold and $\lambda_1 < \ell$, $\ell \in \mathbb{R}$. Then, there exists $w \in H_0^1(\Omega,a)$, with such that $J(w) < 0$ and $\|w\| > \rho$.

Proof

For $t > 0$, consider the function:

$$\phi(t) = J(t\varphi_1) = \frac{t^2}{2}\|\varphi_1\|^2 - \int_\Omega F(x,t\varphi_1)dx.$$

By (V 1) the function $F(x, t) \ge 0$ and the Fatou's Lemma gives:

$$\lim_{t\to+\infty} \frac{1}{t^2}\phi(t) \le \frac{1}{2}\|\varphi_1\|^2 - \int_\Omega \lim_{t\to\infty} \frac{F(x,t\varphi_1)}{(t\varphi_1)^2}\varphi_1^2 dx. \tag{3.4}$$

It follows from (V 3) that:

$$\lim_{t\to+\infty} \frac{1}{t^2}\phi(t) \le \frac{1}{2}\|\varphi_1\|^2 - \frac{\ell}{2}\int_\Omega \varphi_1^2 dx.$$

So:

$$\lim_{t\to+\infty} \frac{1}{t^2}\phi(t) \le \frac{\lambda_1}{2} - \frac{\ell}{2} < 0.$$

We have $\lim_{t\to+\infty} J(t\varphi_1) = -\infty$ and so there exist $w \in H_0^1(\Omega,a)$ with $\|w\| > \rho$ and $J(w) < 0$.

Proof of the Theorem 1.1

(i) Suppose that u is a positive solution or negative solution $\in H_0^1(\Omega,a)$ for (1.1). From the Definition 2.1 and the conditions (V 1) - (V 3) and (V 5), we obtain:

$$\int_\Omega a(x)|\nabla u|^2 dx = \int_\Omega f(x,u)u\,dx \le \int_\Omega \ell u^2 dx \tag{3.5}$$

which gives $\lambda_1 \le \ell$. This finish the proof of (i).

(ii) Let $\lambda_1 < \ell$. If we consider Lemma 3.1 and Lemma 3.2 in mind, we have only to prove the (*PS*) condition for the functional J given by the formula (2.2). For this, consider $\{u_n\}$ a (*PS*) sequence of J:

$$J(u_n) = \frac{1}{2}\|u_n\|^2 - \int_\Omega F(x,u_n)dx, \tag{3.6}$$

$$J(u_n) \to d \textit{ as } n \to +\infty \tag{3.7}$$

for some $d \in \mathbb{R}$ and:

$$\|J'(u_n)\|_* \to 0 \textit{ as } n \to +\infty. \tag{3.8}$$

Step 1 (If $\{u_n\}$ is bounded in $H_0^1(\Omega,a)$, then $\{u_n\}$ is relatively compact)

Suppose that $\{u_n\}$ is bounded in $H_0^1(\Omega,a)$. By compact embedding result, we get up to subsequence:

$$\begin{aligned} &u_n \rightharpoonup u, \textit{ converges weakly in } H_0^1(\Omega,a),\\ &u_n \to u, \textit{ converges strongly in } L^2(\Omega)\\ &\textit{besides}\\ &u_n \to u, \quad \textit{a.e. in } \Omega. \end{aligned}$$

where, $\rightharpoonup$ denotes the weak convergence. From (3.8), we obtain:

$$< J'(u_n), u_n > = \|u_n\|^2 - \int_\Omega f(x,u_n)u_n dx \to 0 \tag{3.9}$$

and also, for all $\varphi \in H_0^1(\Omega,a)$:

$$\int_\Omega a(x)\nabla u_n.\nabla\varphi dx - \int_\Omega f(x,u_n)\varphi dx \to 0. \tag{3.10}$$

So:

$$-div\big(a(x)\nabla u_n\big) - f(x,u_n) \to 0 \ \ in\ H_0^{-1}(\Omega,a), \tag{3.11}$$

where, $H_0^{-1}(\Omega,a)$ indicates the dual space of $H_0^1(\Omega,a)$. From (V 1) - (V 3), $f(x, u_n) \to f(x, u)$ in $L^2(\Omega)$ and so:

$$-div\big(a(x)\nabla u_n\big) \to f(x,u) \ \ in\ H_0^{-1}(\Omega,a). \tag{3.12}$$

As in (Meyers, 1963), we prove that the operator $L = -div(a(x)\nabla)$ is an isomorphism between $H_0^1(\Omega,a)$ and $H_0^{-1}(\Omega,a)$ so:

$$u_n \to L^{-1}\big(f(x,u)\big) \ \ in\ \ H_0^1(\Omega,a). \tag{3.13}$$

Step 2 ($\{u_n\}$ is bounded in $H_0^1(\Omega,a)$)

By contradiction. We suppose that the sequence $\{u_n\}$ is unbounded. Up to subsequence, we have:

$$\|u_n\| \to +\infty \ \ as\ \ n \to +\infty.$$

Let:

$$w_n = \frac{u_n}{\|u_n\|}, t_n = \|u_n\|.$$

The sequence $\{w_n\}$ verifies:

$$\begin{aligned} &w_n \ \ converges\ weakly\ to\ w\ in\ H_0^1(\Omega,a) \\ &w_n \ \ converges\ strongly\ to\ w\ in\ L^2(\Omega) \end{aligned}$$

and also:

$$w_n \ \ converges\ to\ w\ a.e\ in\ \Omega$$

for some w in the space $H_0^1(\Omega,a)$. From the condition (V 3), there exists $C_2 > 0$ such that:

$$\frac{f(x,t)}{t} \le C_2, \ \ for\ all\ t \ne 0,\ and\ \ x \in \Omega \tag{3.14}$$

so:

$$\int_\Omega \frac{f(x,t)}{\|u_n\|^2} u_n dx = \int_\Omega \frac{f(x,u_n)}{u_n} w_n^2 dx \le C\int_\Omega w_n^2 dx. \tag{3.15}$$

From (3.9), the left hand side of (3.15) converge to 1 and then:

$$w \not\equiv 0$$

By using (3.10), we get:

$$\begin{aligned} &\int_\Omega a(x)\nabla w_n.\nabla\varphi dx \\ &-\int_\Omega \frac{f(x,u_n)}{u_n} w_n\varphi dx \to 0 \ \ for\ all\ \varphi \in H_0^1(\Omega,a). \end{aligned} \tag{3.16}$$

By using step 1, we know that:

$$\int_\Omega a(\omega)\nabla w_n.\nabla\varphi d\omega \to \int_\Omega a(\omega)\nabla\omega.\nabla\varphi d\omega, \tag{3.17}$$

for all $\varphi \in H_0^1(\Omega,a)$. Since $u_n(x) = \|u_n\|w_n(x)$, we deduce that $\lim_{n\to+\infty} u_n(x) = \pm\infty$, whenever $w(x) \ne 0$. Let:

$$g_n(x) = \begin{cases} \dfrac{f(x,u_n(x))}{u_n(x)} & if\ \ u_n(x) \ne 0 \\ 0\ if & u_n(x) = 0 \end{cases}$$

From (3.14), the sequence $\{g_n\}$ is bounded on Ω and so it is weakly star convergent in $L^\infty(\Omega)$, up to subsequence to a function g.

By (V 3) and the fact that $u_n(x)$ not equal to zero a.e. in Ω, the function $g(x) = \ell$, for a.e $x \in \Omega$. Now, if we consider the second term of (3.16), we have:

$$\int_\Omega \frac{f(x,u_n)}{u_n} w_n\varphi dx = \int_\Omega g_n(x) w_n\varphi dx$$

That is:

$$\int_\Omega \frac{f(x,u_n)}{u_n} w_n\varphi dx \to \ell \int_\Omega w\varphi dx. \tag{3.18}$$

From (3.16), (3.17) and (3.18), we have:

$$\begin{cases} -div\big(a(x)\nabla w\big) = \ell w \ \ in\ \Omega \\ \qquad\qquad w = 0 \ \ in\ \partial\Omega, \end{cases} \tag{3.19}$$

hence $w = c\varphi_1$ and $\ell = \lambda_1$. This is impossible since we have supposed that $\lambda_1 < \ell < \infty$. We deduce $\{u_n\}$ is bounded in $H_0^1(\Omega,a)$.

Step 3 (Conclusion)

By Proposition 2.1, the Eq. (1.1) has a nontrivial solution.

(iii) Suppose that $\ell = \lambda_1$. Set u a positive solution for (1.1) or negative solution. If we take $\varphi = \varphi_1$ in (2.3), we get:

$$\int_\Omega a\nabla u.\nabla\varphi_1 = \int_\Omega f(x,u)\varphi_1. \tag{3.20}$$

Consider the Eq. (2.4), multiply it by u and integrate, we obtain:

$$\int_\Omega a(x)\nabla u.\nabla\varphi_1\,dx = \ell\int_\Omega u\varphi_1\,dx. \tag{3.21}$$

From (3.20) and (3.21), we get:

$$\int_\Omega \left(f(x,u) - \ell u\right)\varphi_1\,dx = 0.$$

From (V 3) and (V 5), the constant C in the inequality (3.14) will be equal to ℓ and since $\varphi_1 > 0$, we obtain $f(x, u) = \ell u$ a.e. in Ω. This means $f(x, u) = \lambda_1 u$ and then u is an eigenfunction associated to the simple eigenvalue λ_1, so $u = c\varphi_1$ for some constant $c > 0$ or $c < 0$ according to the sign of u.

Conversely, if $\ell = \lambda_1$, $u = c\varphi_1$ for some constant $c \neq 0$ and $f(x, u) = \lambda_1 u$. Then, u is a solution of (1.1).

Proof of the Theorem 1.2

First, we begin by proving the geometric properties.

Lemma 4.1

Assume that (V 1), (V 2), (V 4) hold and $\ell = +\infty$. Then, there exist positive numbers α, ρ verifying:

$$J(u) > \alpha, \forall u \in H_0^1(\Omega, a); \|u\| = \rho.$$

Proof

Using (V 4), there exist $C_2 > 0$ and $t_0 \geq 1$ such that for all $|t| \geq t_0$, $|f(x, t)| \leq C_2|t|^{r-1}$. As in Lemma 3.1 and by using (V 2), we can find $\varepsilon_0 \in (0, 1)$ such that:

$$F(x,t) \leq \frac{1}{2}\lambda_1\left(1-\varepsilon_0\right)t^2 + C_2|t|^r,\ \forall (x,t)\in\Omega\times\mathbb{R}. \tag{4.1}$$

Therefore:

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_1\left(1-\varepsilon_0\right)\|u\|_2^2 - C_2\|u\|_r^r.$$

From the definition of λ_1 and the choice of r, we get:

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{1}{2}\lambda_1\left(1-\varepsilon_0\right)\|u\|^2 - C_2\|u\|^r,$$

that is:

$$J(u) \geq \frac{1}{2}\varepsilon_0\|u\|^2 - C\|u\|^r. \tag{4.2}$$

In the inequality (4.2), we choose $\|u\| = \rho > 0$ small enough, we get $J(u) \geq \alpha$ for some $\alpha > 0$ since $2 < r$.

Now, we pass to the second geometric property of the Proposition 2.1.

Lemma 4.2

Suppose that the function f satisfies (V 1), (V 3), (V 4) and (V 5) with $\ell = +\infty$. Then, $I(t\varphi_1) \to -\infty$ as $t \to +\infty$, where φ_1 is a normalised positive eigenfunction associated to λ_1.

Proof

We have:

$$J(t\varphi_1) = \frac{t^2}{2}\|\varphi_1\|^2 - \int_\Omega F(x,t\varphi_1)\,dx,$$

that is:

$$J(t\varphi_1) = \frac{t^2}{2}\lambda_1 - \int_\Omega F(x,t\varphi_1)\,dx. \tag{4.3}$$

From (V 5), we deduce:

$$0 \leq 2F(x,t) \leq tf(x,t). \tag{4.4}$$

for all $(x, t) \in \Omega\times\mathbb{R}$ and so, for a fixed x the function $\frac{F(x,t)}{t^2}$ is nondecreasing. Therefore, it follows from (V 3) that:

$$\frac{F(x,t)}{t^2} \to +\infty.$$

So, there exists $b > \lambda_1$ and a constant $C_3 > 0$ such that $F(x, t) \geq \frac{b}{2}t^2 + C_3$ for all $t > 0$. The equality (4.3) gives

$$J(t\varphi_1) \leq \frac{t^2}{2}\lambda_1 - \frac{b}{2}t^2\|\varphi_1\|_2^2 - C_3|\Omega|.$$

Then:

$$J(t\varphi_1) \leq t^2\frac{\lambda_1 - b}{2} < 0.$$

So, the Lemma 4.2 is proved.

Similar to the proof of the Lemma 2.3 in (Zhou, 2002), we can prove the following result.

Lemma 4.3

Assume that the condition (V 5) holds and $\|u_n\|$ is a sequence in $H_0^1(\Omega,a)$ such that $< J'(u_n), u_n >\to 0$. Then, up to a subsequence:

$$J(tu_n) \le \frac{1+t^2}{2n} + J(u_n),\ \forall t > 0. \tag{4.5}$$

Proof of the Theorem 1.2

Now, we suppose that $f(x, t)$ is superlinear and it is subcritical at $+\infty$. Let us prove that the problem (1.1) has a nontrivial positive solution.

In order to get the result from Proposition 2.1 and with Lemma 4.1 and Lemma 4.2 in mind, we have to prove that the functional J satisfies compactness property.

Let $\{u_n\}$ a (PS) sequence in $H_0^1(\Omega,a)$ at level d, this means that it satisfying the conditions (3.7) and (3.8). Following the same scheme of the proof of Theorem 1.1 result (ii), we have to prove that the sequence $\{u_n\}$ is bounded in $H_0^1(\Omega,a)$ and the theorem follows. Suppose that $\{u_n\}$ is not bounded and consider:

$$w_n = \frac{u_n}{c\|u_n\|} \text{ and } s_n = c\|u_n\|, \tag{4.6}$$

where $c > 0$ a positive real number.

It is clear that the sequence (w_n) is bounded in $H_0^1(\Omega,a)$, hence there exists $w \in H_0^1(\Omega,a)$ such that, we have the following properties:

$$w_n \rightharpoonup w \text{ in } H_0^1(\Omega,a) \text{ as } n\to+\infty,$$
$$w_n \to w \text{ in } L^2(\Omega) \text{ as } n\to+\infty,$$

up to subsequence and also:

$$w_n(x) \text{ converges to } w(x) \text{ for a.e } x \text{ in } \Omega.$$

Therefore:

$$w_n^+(x) \text{ converges to } w^+(x) \text{ a.e. in } \Omega,$$

and also:

$$w_n^+ \to w^+ \text{ in } L^2(\Omega), \tag{4.7}$$

since for any function $v\in L^2(\Omega)$:

$$v^+ = \frac{v+|v|}{2}.$$

Now, let $\Omega_+ = \{x\in\Omega;\ w^+(x) > 0\}$.

From the fact that $u_n(x) = c\|u_n\|w_n(x)$, we get $u_n^+(x) \to +\infty$ a.e. in Ω_+. By exploiting (4.7) and $\ell = +\infty$, we obtain that for all $B > 0$, there exists n_0, for all $n \ge n_0$:

$$\frac{f(x,u_n^+(x))}{u_n^+(x)}\left(w_n^+(x)\right)^2 \ge B\left(w^+(x)\right)^2. \tag{4.8}$$

It follows from (3.9) that:

$$\begin{aligned}\frac{1}{c^2} &= \lim_{n\to+\infty}\|w_n\|^2 = \lim_{n\to+\infty}\int_\Omega \frac{f(x,u_n)}{u_n}(w_n)^2\,dx\\ &\ge \lim_{n\to+\infty}\int_{\Omega_+}\frac{f(x,u_n^+)}{u_n^+}(w_n^+)^2\,dx\\ &\ge \int_{\Omega_+}\lim_{n\to+\infty}\frac{f(x,u_n^+)}{u_n^+}(w_n^+)^2\,dx.\end{aligned}$$

We deduce by that:

$$\frac{1}{c^2} \ge B\int_{\Omega_+}(w^+)^2\,dx,$$

for all $B > 0$ and so $mes(\Omega_+) = |\Omega_+| = 0$ where mes is the Lebesgue measure. Then:

$$w^+ \equiv 0 \text{ in } \Omega.$$

Therefore, $\lim_{n\to+\infty}\int_\Omega F(x,w_n^+(x))dx = 0$ and we get:

$$J(w_n) \to \frac{1}{2c^2} \text{ as } n\to+\infty. \tag{4.9}$$

If we apply Lemma 4.3, we have up to subsequence:

$$J(w_n) = J(t_n u_n) \le \frac{1}{2n}\left(1+t_n^2\right) + J(u_n). \tag{4.10}$$

where, $t_n = \dfrac{1}{c\|u_n\|}$. From (4.10), (4.9) and (3.7) we get $\dfrac{1}{2c^2} \le d$, this is for any $c > 0$.

So, the Theorem 1.2 is proved. 2

The Theorem 1.2 holds also when the function $f(x, t)$ is superlinear and subcritical at $-\infty$ and the proof is the same.

Conclusion

A weighted elliptic problem with indefinite asymptotically linear nonlinearity is investigated in the present paper. Under suitable conditions, we prove the existence or the nonexistence of nontrivial solutions. Then, we consider the same problem when the nonlinearity is super-linear and in the same time subcritical and we prove an existence result. We use variational method in the proof of the existence of such solutions without using the Ambrosetti-Rabionowitz condition (AR) or any other condition of the same type.

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