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Statistical Inference on a Black-Scholes Model with Jumps. Application in Hydrology

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Abstract: We consider a Stochastic Differential Equation (SDE) driven by a Wiener process and a Poisson measure. This latter measure is associated with a sequence of identically distributed jump amplitudes. Properties of the SDE solution are presented with respect to the associated Wiener and Poisson processes. An algorithm is provided allowing exact numerical simulations of such SDE and implementable within R environment. Statistical inference tools are presented and applied to hydrology data.

Keywords: Stochastic Differential Equation, Wiener Process, Poisson Process, Likelihood Technique

Introduction

In different fields, scientists are confronted with the study of random phenomena. For that purpose, some mathematicians use Stochastic Differential Equations (SDE) to model the random trajectories of these phenomena. They are used in domains such as physics (Calif, 2012), population dynamics (Lungu and Oksendal, 1997), financial mathematics (Black and Scholes, 1973) and biology (Wilkinson, 2011). For instance, in financial mathematics, the Black-Scholes model (1973) is used to describe the volatility of certain options. It is considered as a fundamental step forward for modern finance (Khaled and Samia, 2010). We can also cite stochastic delay Lotka-Volterra model (Bao and Yuan, 2012; Bahar and Mao, 2004) for population dynamics in environmental noise, and processes with jump (Bao *et al*., 2011) as alternative models for phenomena including shocks occurring at random dates associated with random amplitudes. In this paper, we consider a SDE with jumps driven by a Wiener process and a Poisson measure. The solution of this SDE is a stochastic process following a Black-Scholes model with random jump amplitudes. We study the behaviour of this process under mild conditions on the amplitude distribution. Then, we develop the statistical inference about the model parameters (Lacus, 2008) using likelihood techniques (Lo, 1988). Hydrological data are used as an example of application.

Materials and Methods

Black-Scholes Model with Jumps

We consider the Black-Scholes model with jumps. This stochastic process assumes that the solution is determined by the stochastic differential equation:

$$dX_t = -\tau X_t\,dt + \sigma X_t\,dB_t + A_t X_t\,dN_t \quad (1)$$

where τ and σ are given constants. The parameter τ may be regarded as intrinsic rate of decrease, σ is the standard deviation associated with the Brownian term, (B_t) is a standard one-dimensional Brownian motion (Osborne, 1959) and (N_t) a Poisson process. A_t is the jump amplitude at time t and is a positive random variable whose distribution is parameterized by vector θ.

The drift, diffusion, and shock terms on the right side of equation (1) are bounded continuous functions defined on $\mathbb{R}$. Under the conditions of existence and uniqueness of the solution, equation (1) admits a unique positive solution X_t given by:

$$X_t = X_0 exp\left(-\left(\tau + \frac{\sigma^2}{2}\right)t + \sigma B_t + \int_0^t \log\left(1 + A_s\right)dN_s\right).$$

It is worth noticing that equation (1) can be associated with a deterministic model governed by the following equation:

$$dm_t = -\tau m_t\,dt + \lambda a m_t\,dt \quad (2)$$

where λ>0 is the intensity of the Poisson process (N_t) and $a = E(A_t)$ is the expected jump amplitude. Equation (2) is derived from (1) by taking the expectation with respect to (N_t), (B_t) and (A_t). The solution of (2) is

$$m_t = m_0 e_0^{(-\tau+\lambda a)t}.$$

We can notice that the solution explodes when t tends to infinity if the expected jump amplitude $a\lambda > \tau$, but converges to zero if $a\lambda < \tau$. Otherwise, the solution is constant and equal to the initial value m_0. Furthermore, under the condition of independence of the A_t, the expectation and variance are as follows: $E(X_t) = m_t$ and $V(X_t) = m_t^2\left(e^{(\sigma^2+\lambda(b+a^2))t} - 1\right)$ where $b = V(A_t)$ is the amplitude variance. Therefore, X_t converges to zero in probability when $a\lambda + \frac{\sigma^2 + \lambda(b+a^2)}{2} < \tau$.

Distributions Associated with the Solution

X_t conditionally to $(A_j)_{j=1,\dots,N_t}$ and N_t has a log-gaussian distribution with parameters log (K_t) and $\sigma^2 t$, where:

$$K_t = X_0 e^{-\left(\tau+\frac{\sigma^2}{2}\right)t} \times \prod_{j=1}^{N_t}\left(1+A_j\right).$$

Let us write $Y_t = \log(X_t)$. Since $B_t \sim N(0,t)$, this implies that $Y_t \sim N(\log K_t, \sigma^2 t)$ conditionally to K_t.

Let $(X_{t_1}, X_{t_2}, \dots, X_{t_n})$ be the observations of process (X_t), at times $t_1<t_2<\dots<t_n$ in $[0, t]$. The distribution of process (X_t) depends on parameters τ, σ^2, θ and λ which are to be estimated.

Maximum Likelihood Method

The likelihood of $(X_{t_1}, \dots, X_{t_n})$, where n is the number of observation dates, is associated with the likelihood of $Y_t^* = (Y_{t_1}, Y_{t_2} - Y_{t_1}, \dots, Y_{t_n} - Y_{t_{n-1}})$. It is worth noticing that process (Y_t) has independent increments. In fact, for any couple (t_{i-1}, t_i), the increment between these two dates, denoted by $\Delta Y_{t_i} = Y_{t_i} - Y_{t_{i-1}}$, verifies:

$$\Delta Y_{t_i} = -\left(\tau + \frac{\sigma^2}{2}\right)\Delta t_i + \sigma\left(B_{t_i} + B_{t_i-1}\right) + \phi_i$$

where, $t_i - t_{i-1} = \Delta t_i \; and \, \phi_i = \sum_{j=N_{t_{i-1}+1}}^{N_{t_i}} \log\left(1+A_j\right)$. So that the joint likelihood of $\left(Y_t^*, N_t, (A_j)\right)$ is written

$$L\left(\Delta Y_{t_i}; \tau,\sigma,\theta,\lambda\right) = e^{-\lambda t}\frac{(\lambda t)^{N_t}}{N_t!}\prod_{j=1}^{N_t} f_\theta\left(A_j\right) \times \prod_{i=1}^{n}\frac{\exp\left(\frac{-\left(\Delta Y_{t_i} + \left(\tau+\frac{\sigma^2}{2}\right)\left(\Delta_{t_i} - \phi_i\right)\right)^2}{2\sigma^2\Delta_{t_i}}\right)}{\sqrt{2\pi\sigma^2\Delta_{t_i}}}$$

where f_θ is the distribution density of A_j.

Applying the maximum likelihood method, we get the following estimators:

$$\hat{\tau} = \frac{1}{t_n}\left(\sum_{j=1}^{N_t}\log\left(1+A_j\right) - y_{t_n}\right) - \frac{\widehat{\sigma^2}}{2}. \tag{3}$$

Let us write $r^* = \hat{\tau} - \frac{\widehat{\sigma^2}}{2}$:

$$\widehat{\sigma^2} = \frac{\sum_{i=1}^{n}\left(\left(\Delta Y_{t_i} + r^*\Delta t_i - \phi_i\right)^2\left(\Delta t_i\right)^{-1}\right)}{\sum_{i=1}^{n}\left(\Delta Y_{t_i} + r^*\Delta_{t_i} - \phi_i\right)^2} \tag{4}$$

$$\hat{\lambda} = \frac{N_t}{t} \tag{5}$$

In the case where A_j follows the log-normal distribution with parameter $\theta = (\mu, v)$, then

$$\hat{\mu} = \frac{1}{n}\sum_{j=1}^{N_t}\log A_j \tag{6}$$

$$\hat{v} = \frac{1}{n}\sum_{j=1}^{N_t}\left(\log A_j - \hat{\mu}\right)^2 \tag{7}$$

Numerical simulation of the SDE solution

We carried out numerical simulations of the SDE solution by means of an exact method which consists of a three-step algorithm (Appendix):

1. Simulation of the number of Poisson jumps
2. Simulation of dates and jump amplitudes
3. Simulation of classical Black-Scholes model between two consecutive jumps

We were able to build artificial datasets using the following R native functions: rpois, runif, rlnorm, rnorm (Fig. 3).

Results

Application to Hydrological Data

We consider a catalogue of hydrological data from Guadeloupe French West Indies for the period between 5 March 2018 and 25 March 2018. A total of 296 observations were recorded in the HYDRO bank catalogue. This study was carried out according to 5 variables: Station, date, time, water quantity per m^3/s. The water flow is represented in Fig. 1, whereas the water flow difference between two consecutive dates is in Fig. 2.

The estimate of τ, σ^2, λ, μ, v and their Standard Error of Estimate (SEE) are given in Table 1. The *p*-value of the log-likelihood ratio test of nullity for each parameter is also given. The *p*-values are very significant, except for parameter τ which is still significantly non null but at a lower level. The 95% confidence interval for τ, rate of decrease in water flow between Poisson events is [0.024, 0.276]. According to such values for the estimated rate of decrease b τ, the convergence in probability of (Xt) to zero is not verified which means that drying out does not occur at the station under study.

Application to Artificial Data

Based on the results obtained from the hydrology data, we carried out numerical simulations with a set of parameters similar to the estimate values of Table 1. Figure 3 shows an example of such trajectories for the solution of Equation 1. For each simulated trajectory, the maximum likelihood method provided parameter estimates. Therefore, from the whole set of trajectory simulations, we could get sample distribution of the maximum likelihood estimator for each parameter. The classical properties of unbiasedness and normality were then checked.

Table 1: Parameter estimation and nullity test results for the water flow data

Estimator	Estimate	SEE	*p*-value
$\hat{\tau}$	0.1499	0.0643	0.0197
$\widehat{\sigma^2}$	0.0700	0.0092	0.0000
$\hat{\mu}$	-1.9524	0.1093	0.0000
$\hat{\nu}$	0.4765	0.0773	0.0000
$\hat{\lambda}$	1.1176	0.2564	0.0000

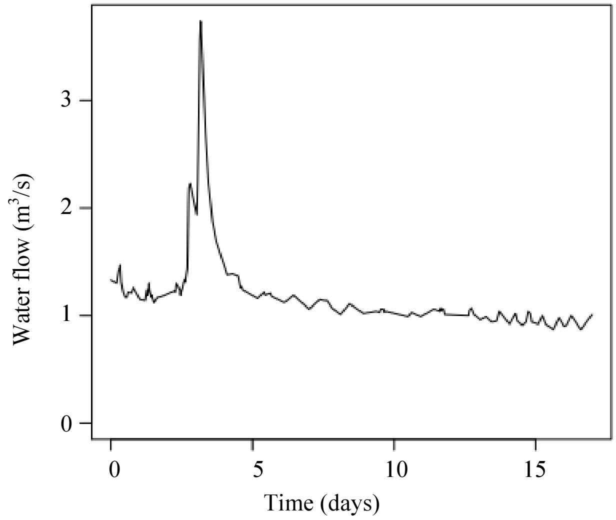

Fig. 1: Water flow distribution between the 5th and 25th of March, 2018, from a station of Guadeloupe, F.W.I

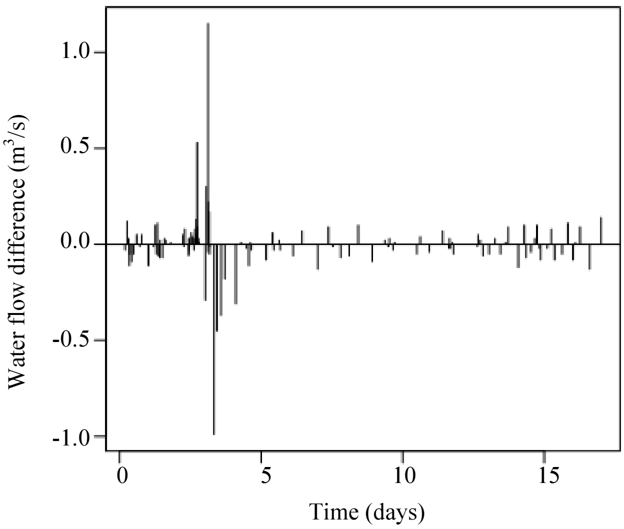

Fig. 2: Water flow difference between two consecutive dates, for the same data as in Figure 1

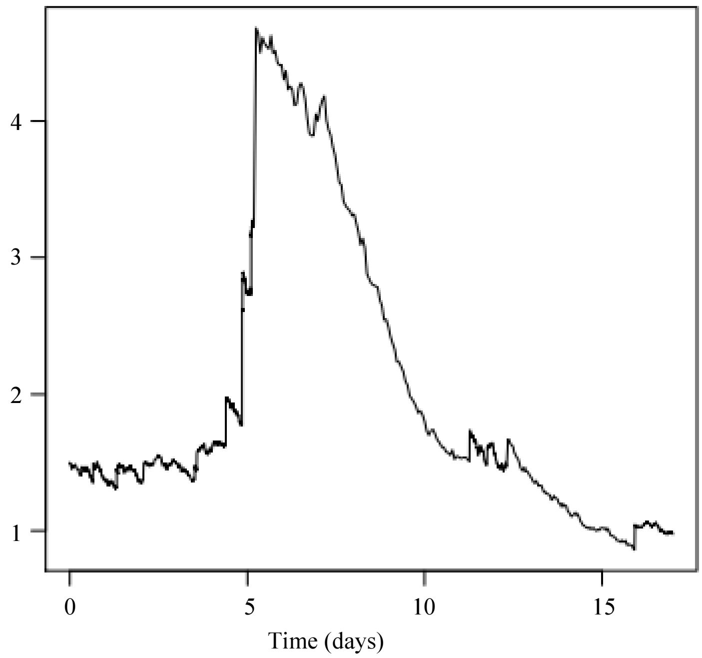

Fig. 3: Simulated solution of equation (1) for $(\tau, \sigma^2, \lambda, \mu, \nu) = (0.15, 0.07, 1.00, -2.00, 0.5)$

Discussion

We considered a continuous time stochastic process $X = (X_t)$ which is solution of a SDE associated with the Black-Scholes model with jumps. Under the assumption of independence and equality of expectations and variances for the jump amplitudes, we gave conditions on the model parameters for convergence in probability of (X_t) to zero. It would be interesting to see how to weaken the assumptions on the jump amplitude process (A_t) to get convergence results. The statistical inference about this model was developed for observations of X at n dates and observations of time and amplitude of jumps over a time windows $[0, t]$. It would be interesting to treat the case for which jump times and jump amplitudes are not available.

Conclusion

In this study, we have presented a SDE driven by a Wiener process and a Poisson measure whose solution follows a Black-Scholes model with jumps. Under independence and stationarity assumptions on the jump amplitude process, we get convergence in probability for the stochastic process solution of this SDE. The solution can be numerically simulated in R programming environment. From observations of the process at different dates, as well as those of jump times and amplitudes, likelihood techniques can be implemented and provide statistical inference tools. As an illustration, we used data collected by the HYDRO bank on water level measurements in Guadeloupe, French West Indies.

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Author's Contributions

All authors equally contributed in this work.

Ethics

The authors declare that there is no conflict of interests regarding the publication of this article which is original and contains unpublished material.

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Appendix:

R script for numerical simulations of the SDE.

```
sim2=function(X0,tau,sigma,lambda,mu,nu,t,MaxY){
#Simulation of the SDE : dXt=Xt(-tau*dt+sigma*dBt+At*dNt)
#X0 is the initial value ; X0>0
#tau is the rate of decrease
#sigma is the standard deviation of the Wiener process
#lambda is the Poisson process intensity ; lambda>0
#t is the experiment duration ; t>0
n=rpois(1,lambda*t)
dates=c(sort(runif(n,max=t)))
sauts=rlnorm(n,meanlog=mu,sdlog=sqrt(nu))
r=tau+(sigma^2)/2
# simulation of Wiener process between 0 and first jump time
Brown=cumsum(c(0,rnorm(100,mean=0,sd=sqrt(dates[1]/100))))
#Simulation of processus between 0 and first jump time
valeurs=curve(X0*exp(-r*x+sigma*Brown),from=0,to=dates[1],add=TRUE,type=”n”)$y
datejours=seq(0,dates[1],length.out=101)
debit=valeurs
xdates=c(dates,t)
for(i in 1:n){
lines(rep(dates[i],2),c(valeurs[101],valeurs[101]*(1+sauts[i])))
X1=valeurs[101]*(1+sauts[i]) #Initial condition modification
#Simulation of Wiener process between two consecutive jumps
Brown=cumsum(c(0,rnorm(100,mean=0,sd=sqrt((xdates[i+1]-xdates[i])/100))))
#Simulation of process between two consecutive jumps
valeurs=curve(X1*exp(-r*(dates[i])+sigma*Brown),from=xdates[i],to=xdates[i+1],add=TRUE,type=”n”)$y
datejours=c(datejours,seq(xdates[i],xdates[i+1],length.out=102)[-1])
debit=c(debit,valeurs)
}
B=rep(0,length(datejours))
B[seq(101,101*n,101)]=log(1+sauts) # value of log(1+A(ti))
jeudon=cbind(datejours,debit)
resufinal=list(jeudon,dates,sauts,B)
resufinal
}
```