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Interaction Model in Statistical Mechanics

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Abstract: Statistical mechanics considers several models such as Ising model, Potts model, Heisenberg model etc. A rigorous mathematical approach based on the axiomatic foundation of probability would benefit the study and applications of these models. In this paper we use this approach to generalize some of these models into one construction named an interaction model. We introduce a mathematically rigorous definition of the model on an integer lattice that describes a physical system with many particles interacting with an external force and with one another; a random field X_t $\left(t \in \mathbb{Z}^\nu\right)$ models some property of the system such as electric charge, density etc. We introduce a finite model first and then define the thermodynamic limit of the finite models with Gibbs probability measure. The set of values of X_t can be unbounded for more generality. We study properties of the interaction model and show that Ising and Potts models are particular cases of the interaction model.

Keywords: Infinite Particle System, Gibbs Measure, Radius of Interaction, Thermodynamic Limit, Ising Model, Potts Model

Introduction

Statistical mechanics studies models of physical systems with many particles, which interact with an external force and with one another. Well-known models include Ising, Potts, Heisenberg, and n-vector models (see, for example, Duminil-Copin *et al*., 2017; Kashapov, 1977; Külske *et al*., 2014; Malyshev and Minlos, 1991; Malyshev, 1980). In these models a random field X_t is used to model some property of the system such as electric charge, density etc.

Ising model is the simplest and most popular model. It describes a system with two states and models the phenomenon of ferromagnetism. It is also used in quantum field theory. Potts model is a generalization of Ising model to a system with a finite number of states.

An n-vector model represents classical spins by n-dimensional vectors of unit length. This model can be used to describe many physical phenomena. Particular cases of this model include the Ising model for $n = 1$, XY-model for $n = 2$ and Heisenberg model for $n = 3$.

Kachapova and Kachapov (2016) introduced the concept of interaction model as a generalization of some existing models; there we provided a proof based on this concept that the random field X_t transformed by renormalization group converges to an independent random field with Gaussian distribution.

In this paper we generalize and improve the interaction model from (Kachapova and Kachapov, 2016). The new model does not have restrictions on the distribution of X_t and the set of values of X_t can be unbounded, which is an advantage of this model comparing to all aforementioned models, which have the values of X_t bounded.

In this paper we use a rigorous mathematical approach based on the axiomatic foundation of probability. We introduce a mathematically precise definition of interaction model on an integer lattice: first as a finite model and then as the thermodynamic limit of the finite models with Gibbs probability measure.

We study properties of the interaction model and show how some well-known models are represented as particular cases of the interaction model.

In Section 1 we introduce main components of the interaction model of a physical system that include an integer lattice $\mathbb{Z}^\nu$, the set of configurations of the system, initial independent probability measure P_0 and a random field $X_t (t \in \mathbb{Z}^\nu)$ that models a property of the system. Next we introduce three characteristics of the interaction model: a main parameter λ, radius of interaction r and potential Φ.

In Section 2 we study Gibbs modification of a probability measure. In particular, we split Gibbs modification of the initial independent probability measure into two steps reflecting the influence of an external field on the first step and the interaction between particles on the second step. The first-step modification is mathematically simple and leaves the field X_t independent, therefore this construction simplifies mathematical computations.

In Section 3 we define a finite interaction model on an integer cube using Gibbs modification of the initial probability measure P_0. We prove some properties of the finite model.

In Section 4 we define an infinite interaction model and in Section 5 we show that Ising and Potts models are particular cases of the interaction model. In Section 6 we discuss how to generalize our model, so that the n-vector model becomes its particular case too.

1. Main Components of Interaction Model

In Section 3 we will construct an interaction model to describe a physical system with many particles. Here we introduce its main components.

Definition 1.1

1. Fix a natural number $\nu\geq 1$ and consider a ν-dimensional integer lattice:

$$\mathbb{Z}^{\nu} = \left\{\left(\tau_1,...,\tau_2\right)\middle|\tau_i \in \mathbb{Z}, i = 1,...,\nu\right\}$$

with the distance between any two points $s,\tau \in \mathbb{Z}^{\nu}$ defined by:

$$\left\|s-\tau\right\| = \sum_{i=1}^{\nu}\left|s_i - \tau_i\right|.$$

2. $\Omega = \left\{\omega \mid \omega : \mathbb{Z}^{\nu} \to \mathbb{R}\right\}$.

An element ω of Ω is called a **configuration** and is interpreted as a **state** of the physical system

3. For any $t \in \mathbb{Z}^{\nu}$ a function $X_t : \Omega \to \mathbb{R}$ is defined by the following:

$$X_t\left(\omega\right) = \omega\left(t\right).$$

4. Denote Σ the σ-algebra generated by sets of the form $\{\omega \in \Omega \mid \omega(t) \leq a\}$ for all $t \in \mathbb{Z}^{\nu}, a \in \mathbb{R}$.
5. Fix P_0, a probability measure on (Ω, Σ) such that for any $a_1,...,a_n \in \mathbb{R}$ and distinct $t_1,...,t_n \in \mathbb{Z}^{\nu}$:

$$P_0\left(\bigcap_{i=1}^{n}\left\{\omega(t_i) \leq a_i\right\}\right) = \prod_{i=1}^{n} P_0\left(\omega(t_i) \leq a_i\right). \quad (1)$$

We call P_0 the **initial probability measure**.

For the rest of the paper we fix the objects $\mathbb{Z}^{\nu}, \Omega, X, \Sigma$ and P_0 from this definition.

Remark

There always exists P_0 satisfiying (1). For example, if F is any probability distribution function, we can take $P_0(\omega(t) \leq a) = F(a)$ for any t and define the rest by formula (1).

Lemma 1.2

1) (Ω, Σ, P_0) is a probability space.
2) $\left\{X_t \mid t \in \mathbb{Z}^{\nu}\right\}$ is an independent random field on this probability space.

Proof

The lemma immediately follows from the definitions. ■

Definition 1.3

1. Consider a **graph** (V, E), where the set of vertices V is a finite subset of $\mathbb{Z}^{\nu}$ and E is the set of edges; each edge can be regarded as a pair of distinct vertices (there are no loops). The length of each edge is the distance between its end vertices.
2. The graph (V, E) is called **1-connected** if it is connected and the length of any of its edges equals 1.
3. For a finite set $B \subset \mathbb{Z}^{\nu}$ define its **size** $S(B)$ as the minimum number of edges of 1-connected graphs (V, E) such that $B \subseteq V$.

Definition 1.4

Here we introduce three main characteristics of interaction: λ, r, Φ and a set $\mathfrak{B}$.

1. $\lambda \in \mathbb{R}, 0 \leq \lambda < 1$. λ is called the **main parameter** of the model.
2. $r \in \mathbb{R}, r \geq 1$. r is called the **radius of interaction**.
3. Denote $\mathfrak{B} = \left\{B \subset \mathbb{Z}^{\nu} \middle| B \neq \varnothing \text{ and size } S\left(B\right) \leq r\right\}$.
4. For each $B \in \mathfrak{B}, \Phi_B$ is a random variable on the probability space (Ω, Σ_B, P_0), where Σ_B is the σ-algebra generated by sets of the form $\left\{\omega \in \Omega \mid \omega(t) \leq a\right\}, t \in B, a \in \mathbb{R}$; all Φ_B satisfy the condition:

$$\left|\Phi_B\right| \leq \lambda^{S(B)}.$$

Φ is called the **potential** of the system.

Clearly, each set $B\in\mathcal{B}$ is finite. Φ_B characterizes the interaction energy of the set B. If B consists of two or more points, then the random variable Φ_B represents interaction between elements of the set B. If $B = \{t\}$ is a singleton, then Φ_B represents interaction of t with an external field and the influence of kinetic energy.

Lemma 1.5

If sets B, $C\in\mathcal{B}$ and $B\cap C = \varnothing$, then the random variables Φ_B and Φ_C are independent with respect to the probability measure P_0, that is on the probability space (Ω, Σ, P_0).

Proof

This is proven using standard techniques of probability theory, first for the case of discrete Φ_B and Φ_C, and then for general random variables Φ_B, Φ_C as limits of discrete random variables. ■

2. Gibbs Modification

To describe interaction between particles, we modify the initial probability measure P_0, so that the corresponding distribution of the random field $\{X_t \mid t\in\mathbb{Z}^\nu\}$ is not independent any more. This section describes the modification in general.

For any probability measure P on (Ω, Σ) denote $\langle . \rangle_P$ the expectation with respect to P.

Definition 2.1

Suppose P is a probability measure on (Ω, Σ) and U is a bounded random variable on (Ω, Σ).

Gibbs modification of the probability measure P by the random variable U is denoted P_U and is defined as follows. For any event $A\in\Sigma$:

$$P_U(A) = \frac{\langle I_A e^U\rangle_P}{\langle e^U\rangle_P}, \tag{2}$$

where I_A denotes the indicator of event A.

Remark

Since the random variable U is bounded, both expectations in formula (2) exist and $\langle e^U\rangle_P > 0$. So $P_U(A)$ is always defined.

The following lemma is used in literature without proof. In order to have a complete picture we provide an accurate proof here.

Lemma 2.2

In conditions of the previous definition:

1) P_U is a probability measure on (Ω, Σ);

2) for any random variable Y on (Ω,Σ), $\langle Y\rangle_{P_U} = \dfrac{\langle Ye^U\rangle_P}{\langle e^U\rangle_P}$.

Proof

1)

$$P_U(\varnothing) = \frac{\langle I_\varnothing e^U\rangle_P}{\langle e^U\rangle_P} = \frac{0}{\langle e^U\rangle_P} = 0;$$

$$P_U(\Omega) = \frac{\langle I_\Omega e^U\rangle_P}{\langle e^U\rangle_P} = \frac{\langle e^U\rangle_P}{\langle e^U\rangle_P} = 1.$$

To complete the proof it remains to show that for any sequence of disjoint events $A_i\in\Sigma$ $(i = 1, 2, ...)$ the following holds:

$$P_U\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty P_U(A_i). \tag{3}$$

Denote $A = \bigcup_{i=1}^\infty A_i$. First we prove:

$$\langle I_A e^U\rangle_P = \sum_{i=1}^\infty \langle I_{A_i} e^U\rangle_P. \tag{4}$$

U is a bounded random variable, so for some constant M, $|U|\le M$ and $0 < e^U \le e^M$.

Since P is a probability measure, we have:

$$P(A) = P\left(\bigcup_{i=1}^\infty A_i\right) = \sum_{i=1}^\infty P(A_i).$$

So for any $\varepsilon>0$ there is $n\in\mathbb{N}$ such that $\sum_{i=n+1}^\infty P(A_i)\le \varepsilon e^{-M}$. Denote $B_n = \bigcup_{i=n+1}^\infty A_i$. Then:

$$P(B_n)\le \varepsilon e^{-M}. \tag{5}$$

$$A = \bigcup_{i=1}^n A_i \cup B_n, \quad I_A = \sum_{i=1}^n I_{A_i} + I_{B_n}.$$

$$\langle I_A e^U\rangle_P = \left\langle\left(\sum_{i=1}^n I_{A_i} + I_{B_n}\right)e^U\right\rangle_P = \left\langle \sum_{i=1}^n I_{A_i} e^U + I_{B_n} e^U\right\rangle_P$$
$$= \sum_{i=1}^n \langle I_{A_i} e^U\rangle_P + \langle I_{B_n} e^U\rangle_P.$$

$$0 \le \langle I_A e^U\rangle_P - \sum_{i=1}^n \langle I_{A_i} e^U\rangle_P = \langle I_{B_n} e^U\rangle_P$$
$$\le \langle I_{B_n} e^M\rangle_P \le e^M \langle I_{B_n}\rangle_P = e^M P(B_n)$$
$$[by (5)] \le e^M \varepsilon e^{-M} = \varepsilon.$$

Thus:

$$\left|\sum_{i=1}^{n}\left\langle I_{A_i} e^U\right\rangle_P - \left\langle I_A e^U\right\rangle_P\right| \le \varepsilon.$$

Therefore $\lim_{n\to\infty}\sum_{i=1}^{n}\left\langle I_{A_i} e^U\right\rangle_P = \left\langle I_A e^U\right\rangle_P$. That is, the series $\sum_{i=1}^{\infty}\left\langle I_{A_i} e^U\right\rangle_P$ converges and $\sum_{i=1}^{\infty}\left\langle I_{A_i} e^U\right\rangle_P = \left\langle I_A e^U\right\rangle_P$. (4) is proven.

Proof of (3)

$$P_U\left(\bigcup_{i=1}^{\infty} A_i\right) = P_U(A) = \frac{\left\langle I_A e^U\right\rangle_P}{\left\langle e^U\right\rangle_P} = \left[by(4)\right] = \frac{\sum_{i=1}^{\infty}\left\langle I_{A_i} e^U\right\rangle_P}{\left\langle e^U\right\rangle_P}$$

$$= \sum_{i=1}^{\infty}\frac{\left\langle I_{A_i} e^U\right\rangle_P}{\left\langle e^U\right\rangle_P} = \sum_{i=1}^{\infty} P_U(A_i).$$

2) Consider 3 cases.

- Case 1: $Y = I_A$ (Y is an indicator of some event $A \in \Sigma$)

  Then $\langle Y\rangle_{P_U} = \langle I_A\rangle_{P_U} = P_U(A) = \frac{\langle I_A e^U\rangle_P}{\langle e^U\rangle_P} = \frac{\langle Y e^U\rangle_P}{\langle e^U\rangle_P}$.

- Case 2: $Y = \sum_{i=1}^{n} a_i Y_i$, where each Y_i is an indicator.

  Then:

$$\langle Y\rangle_{P_U} = \sum_{i=1}^{n} a_i \langle Y_i\rangle_{P_U} = [by\ Case\ 1]$$

$$= \sum_{i=1}^{n} a_i \frac{\langle Y_i e^U\rangle_P}{\langle e^U\rangle_P} = \frac{\sum_{i=1}^{n} a_i \langle Y_i e^U\rangle_P}{\langle e^U\rangle_P}$$

$$= \frac{\left\langle \sum_{i=1}^{n} \alpha_i Y_i e^U\right\rangle_P}{\langle e^U\rangle_P} = \frac{\langle Y e^U\rangle_P}{\langle e^U\rangle_P}.$$

- Case 3: general case.

It is a well known fact in probability theory that any random variable can be represented as a uniform limit of discrete random variables. Thus, for any $\omega \in \Omega$:

$$Y(\omega) = \lim_{n\to\infty} V_n(\omega),$$

where the convergence is uniform with respect to $\omega \in \Omega$ and each V_n is a discrete random variable with a finite number of values; this is a random variable from Case 2.

Then:

$$\lim_{n\to\infty}\langle V_n\rangle_{P_U} = \langle Y\rangle_{P_U}. \tag{6}$$

As $n \to \infty$, we have for any $\omega \in \Omega$: $V_n(\omega) - Y(\omega) \to 0$ uniformly on Ω, so $|V_n(\omega) - Y(\omega)| \to 0$ and:

$$\lim_{n\to\infty}\langle | V_n - Y |\rangle_P \to 0 \ \ as \ \ n \to \infty. \tag{7}$$

U is a bounded random variable, so for some constant M, $|U| \le M$ and $0 < e^U \le e^M$.

$$\left|\langle V_n e^U\rangle_P - \langle Y e^U\rangle_P\right| = \left|\langle (V_n - Y) e^U\rangle_P\right|$$

$$\le \left\langle \left|(V_n - Y) e^U\right|\right\rangle_P \le \left\langle e^M |V_n - Y|\right\rangle_P$$

$$= e^M \left\langle |V_n - Y|\right\rangle_P \to 0 \ \ as \ \ n \to \infty \ by \ (7).$$

Therefore:

$$\lim_{n\to\infty}\langle V_n e^U\rangle_P = \langle Y e^U\rangle_P. \tag{8}$$

$$\langle Y\rangle_{P_U} = \left[by(6)\right] = \lim_{n\to\infty}\langle V_n\rangle_{P_U} = [by\ Case\ 2] = \lim_{n\to\infty}\frac{\langle V_n e^U\rangle_P}{\langle e^U\rangle_P}$$

$$= \frac{\lim_{n\to\infty}\langle V_n e^U\rangle_P}{\langle e^U\rangle_P} = \left[by(8)\right] = \frac{\langle Y e^U\rangle_P}{\langle e^U\rangle_P}. \ \blacksquare$$

Theorem 2.3

Suppose P is a probability measure on (Ω, Σ), U_1 and U_2 are bounded random variables on (Ω, Σ) and $U = U_1 + U_2$.

Suppose $P_1 = P_{U_1}$ and $P_2 = (P_1)_{U_2}$ are consecutive Gibbs modifications. Then $P_2 = P_U$.

Proof

Using Lemma 2.2.2), for any event $A \in \Sigma$ we get:

$$P_2(A) = \frac{\langle I_A e^{U_2}\rangle_{P_1}}{\langle e^{U_2}\rangle_{P_1}} = \frac{\dfrac{\langle I_A e^{U_2} e^{U_1}\rangle_P}{\langle e^{U_1}\rangle_P}}{\dfrac{\langle e^{U_2} e^{U_1}\rangle_P}{\langle e^{U_1}\rangle_P}}$$

$$= \frac{\langle I_A e^{U_2} e^{U_1}\rangle_P}{\langle e^{U_2} e^{U_1}\rangle_P} = \frac{\langle I_A e^{U_1 + U_2}\rangle_P}{\langle e^{U_1 + U_2}\rangle_P} = P_U(A). \ \blacksquare$$

3. Finite Interaction Model

Denote **0** = (0,...,0), the origin in $\mathbb{Z}^v$.

Definition 3.1.

A **finite interaction model** with characteristics λ, r, Φ (from Definition 1.4) is a sequence $(\Lambda, X, U_\Lambda, \mathcal{A}_\Lambda)$ of four objects defined as follows.

1. $\Lambda = \{ t \in \mathbb{Z}^\nu \mid \| t - \mathbf{0} \| \le N \}$ for a fixed positive integer N. Thus, Λ is a cube in the lattice $\mathbb{Z}^\nu$.
2. X is the fixed random field on (Ω, Σ) introduced in Definition 1.1.
3. A function $U_\Lambda : \Omega \to \mathbb{R}$ is called the **interaction energy** and is defined by the following:

$$\text{for any } \omega \in \Omega, U_\Lambda(\omega) = \sum_{B \in \mathfrak{B}, B \subset \Lambda} \Phi_B(\omega), \tag{9}$$

where the set $\mathfrak{B}$ is defined in Definition 1.4. $U_\Lambda(\omega)$ characterizes the energy of configuration ω in Λ.
4. Denote $P_\Lambda = (P_0)_{U_\Lambda}$, the Gibbs modification of the initial probability P_0 by the interaction energy U_Λ. $\mathcal{A}_\Lambda$ is the probability space $(\Omega, \Sigma, P_\Lambda)$.

This ends the Definition 3.1.

Remark 1

The random variable U_Λ is bounded because the sum (9) has a finite number of addends and each $|\Phi_B| \le \lambda^{S(B)} < \lambda$, since $0 \le \lambda < 1$. Therefore Lemma 2.2.1) holds when stated for P_Λ instead of $P_U : P_\Lambda$ is a probability measure on (Ω, Σ) and $\mathcal{A}_\Lambda$ is indeed a probability space.

Remark 2

The random field $\{X_t \mid t \in \mathbb{Z}^\nu\}$ is independent on the base probability space (Ω, Σ, P_0) but it may not be independent on the probability space $(\Omega, \Sigma, P_\Lambda)$.

The finite interaction model describes a physical system with many particles represented by points in an integer cube. The random field X_t describes some property of the physical system.

The main parameter λ from Definition 1.4 is positive and characterizes the temperature of the system: low values of λ correspond to high temperatures. Our interaction model describes systems with fairly high temperatures. For $\lambda = 0$ the model describes ideal gas.

The interaction model generalizes some well-known models in statistical mechanics (we give details in Sections 5 and 6). In those models the values of random variables X_t are bounded. Here we have a more general case when the values of X_t are not bounded.

For brevity we denote $\langle . \rangle_{P_0}$ as $\langle . \rangle_0$.

Lemma 3.2.

Suppose $t_1, \ldots, t_n \in \mathbb{Z}^\nu \setminus \Lambda$ and $a_1, \ldots, a_n \in \mathbb{R}$. Then $P_\Lambda[\omega(t_1) \le a_1, \ldots, \omega(t_n) \le a_n] = P_0[\omega(t_1) \le a_1, \ldots, \omega(t_n) \le a_n]$.

Proof

For any $i = 1, 2, \ldots, n$, denote $A_i = \{\omega(t_i) \le a_i\}$ and denote $A = \bigcap_{i=1}^{n} A_i$.

If $t \in \Lambda$ and $a \in \mathbb{R}$, the events A_i and $\{\omega(t) \le a\}$ are independent with respect to the probability measure P_0, that is on the probability space (Ω, Σ, P_0); this follows from the Definition 1.1. Based on that, similarly to Lemma 1.5 it is proven that for any $B \subset \Lambda$ the random variables I_A and Φ_B are independent with respect to P_0. Therefore I_A and $e^{U_\Lambda} = \prod_{B \in \mathfrak{B}, B \subset \Lambda} e^{\Phi_B}$ are independent with respect to P_0. Using this independence we get:

$$P_\Lambda[\omega(t_1) \le a_1, \ldots, \omega(t_n) \le a_n]$$
$$= P_\Lambda(A) = \frac{\langle I_A e^{U_\Lambda} \rangle_0}{\langle e^{U_\Lambda} \rangle_0} = \frac{\langle I_A \rangle_0 \langle e^{U_\Lambda} \rangle_0}{\langle e^{U_\Lambda} \rangle_0}$$
$$= \langle I_A \rangle_0 = P_0(A) = P_0[\omega(t_1) \le a_1, \ldots, \omega(t_n) \le a_n]. \blacksquare$$

For the rest of this section we fix the finite interaction model from Definition 3.1.

Definition 3.3

Here we introduce random variables U' and U''. For any $\omega \in \Omega$ we define:

$$U'(\omega) = \sum_{B \in \mathfrak{B},\, B \subset \Lambda, |B| = 1} \Phi_B(\omega), \quad U''(\omega) = \sum_{B \in \mathfrak{B},\, B \subset \Lambda, |B| > 1} \Phi_B(\omega).$$

Thus, $U = U' + U''$. The function U is split into U' and U'', where U' is a sum over singleton sets B and U'' is a sum over sets B with two or more elements.

Consider consecutive Gibbs modifications $P' = (P_0)_{U'}$ and $P'' = (P')_{U''}$. We call P' the **single modification** and P'' the **plural modification**.

Lemma 3.4.

1) For any $t \in \Lambda$ and $x \in \mathbb{R}$:

$$P'(X_t \le x) = \frac{\langle I_A e^{\Phi_{\{t\}}} \rangle_0}{\langle e^{\Phi_{\{t\}}} \rangle_0},$$

where $A = \{X_t \le x\}$.

2) For any $t \in \mathbb{Z}^\nu \setminus \Lambda$ and $x \in \mathbb{R}$:

$$P'(X_t \le x) = P_0(X_t \le x).$$

3) After the single modification the field $\{X_t \mid t \in \mathbb{Z}^\nu\}$ is still independent. That is, this field is independent on the probability space (Ω, Σ, P′).

Proof

For brevity we will write $\Phi_{\{\tau\}}$ as Φ_τ. Thus:

$$U' = \sum_{\tau\in\Lambda} \Phi_\tau \text{ and } e^{U'} = \prod_{\tau\in\Lambda} e^{\Phi_\tau}. \tag{10}$$

1) $$P'(X_t \le x) = P'(A) = \frac{\langle I_A e^{U'}\rangle_0}{\langle e^{U'}\rangle_0} = [by\ (10)] = \frac{\left\langle I_A \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}$$

$$= \frac{\left\langle (I_A \cdot e^{\Phi_t}) \cdot \left(\prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right)\right\rangle_0}{\left\langle e^{\Phi_t} \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0} =$$ (due to independence with respect to measure P_0 by Lemma 1.5)

$$= \frac{\langle I_A \cdot e^{\Phi_t}\rangle_0 \cdot \left\langle \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0}{\langle e^{\Phi_t}\rangle_0 \cdot \left\langle \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0} = \frac{\langle I_A e^{\Phi_t}\rangle_0}{\langle e^{\Phi_t}\rangle_0}.$$

2) Denote $A = \{X_t \le x\}$.

$$P'(X_t \le x) = P'(A) = \frac{\langle I_A e^{U'}\rangle_0}{\langle e^{U'}\rangle_0} = [\text{by } (10)] = \frac{\left\langle I_A \prod_{\tau\in\Lambda} e^{\Phi\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi\tau}\right\rangle_0} =$$

(due to independence with respect to measure P_0) =

$$= \frac{\langle I_A\rangle_0 \left\langle \prod_{\tau\in\Lambda} e^{\Phi\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi\tau}\right\rangle_0} = \langle I_A\rangle_0 = P_0(A) = P_0(X_t \le x).$$

3) We need to prove that for any $a_1,...,a_n \in \mathbb{R}$ and distinct $t_1,...,t_n \in \mathbb{Z}^\nu$:

$$\begin{aligned} &P'\left(X_{t_1} \le a_1, X_{t_2} \le a_2, ..., X_{t_n} \le a_n\right) \\ &= P'\left(X_{t_1} \le a_1\right) \cdot ... \cdot P'\left(X_{t_n} \le a_n\right). \end{aligned} \tag{11}$$

Denote $t = t_1$, $B = \{X_t \le a_1\}$ and $C = \{X_{t_2} \le a_2, ..., X_{t_n} \le a_n\}$. First we prove:

$$P'(B \cap C) = P'(B)P'(C). \tag{12}$$

Its left-hand side equals:

$$P'(B\cap C) = \frac{\langle I_{B\cap C}\, e^{U'}\rangle_0}{\langle e^{U'}\rangle_0} = [by(10)] = \frac{\left\langle I_B \cdot I_C \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}.$$

Consider 2 cases.

- Case 1: $t \in \Lambda$.
  Then:

$$P'(B\cap C) = \frac{\left\langle (I_B \cdot e^{\Phi_t}) \cdot \left(I_C \prod_{\tau\in\Lambda,\tau\ne t,} e^{\Phi_\tau}\right)\right\rangle_0}{\left\langle e^{\Phi_t} \prod_{\tau\in\Lambda,\tau\ne t,} e^{\Phi_\tau}\right\rangle_0}$$

= (due to independence with respect to measure P_0 by Lemma 1.5):

$$= \frac{\langle I_B \cdot e^{\Phi_t}\rangle_0 \cdot \left\langle I_C \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0}{\langle e^{\Phi_t}\rangle_0 \left\langle \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0} = \frac{\langle I_B \cdot e^{\Phi_t}\rangle_0}{\langle e^{\Phi_t}\rangle_0} \cdot \frac{\left\langle I_C \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda,\tau\ne t} e^{\Phi_\tau}\right\rangle_0} \cdot \frac{\langle e^{\Phi_t}\rangle_0}{\langle e^{\Phi_t}\rangle_0}$$

$= [\text{by part 1})]$

$$= P'(B) \cdot \frac{\left\langle I_C \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0} = P'(B) \cdot \frac{\langle I_C\, e^{U'}\rangle_0}{\langle e^{U'}\rangle_0} = P'(B) \cdot P'(C).$$

- Case 2: $t \in \mathbb{Z}^\nu \setminus \Lambda$.
  Due to independence with respect to measure P_0 we have:

$$P'(B\cap C) = \frac{\left\langle I_B \cdot I_C \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0} = \frac{\langle I_B\rangle_0 \left\langle I_C \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}$$

$$= \langle I_B\rangle_0 \frac{\left\langle I_C \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0}{\left\langle \prod_{\tau\in\Lambda} e^{\Phi_\tau}\right\rangle_0} = P_0(B)P'(C) = P'(B)P'(C)$$

by part 2). Thus, (12) is proven. Then:

$$\begin{aligned} &P'(X_{t_1} \le a_1, X_{t_2} \le a_2, ..., X_{t_n} \le a_n) \\ &= P'(X_{t_1} \le a_1) \cdot P'(X_{t_2} \le a_2, ..., X_{t_n} \le a_n). \end{aligned}$$

Continuing by induction we get (11). ■

Thus, the single modification can change distributions of random variables $X_t \left(t \in \mathbb{Z}^\nu\right)$ but it does not change their independence. Note that we do not put any restriction on initial distributions of X_t.

4. Infinite Interaction Model

Definition 4.1

An **infinite interaction model** with characteristics λ, r, Φ is the ordered sequence of two objects $(X, \mathcal{A})$ defined as follows.

1. X is the fixed random field on (Ω, Σ) introduced in Definition 1.1.
2. For any $N = 1,2,...$ denote $\Lambda_N = \{ t \in \mathbb{Z}^\nu \mid \| t - \mathbf{0} \| \leq N\}$.

   Let $(\Lambda_N, X_{\Lambda_N} \mathcal{A}_{\Lambda_N})$ be the finite interaction model and $P_N = P_{\Lambda_N}$ the corresponding probability measure from Definition 3.1.
   Let P_λ be a probability measure on (Ω, Σ) that is a limit of the probability measures P_N in some sense, for example, their weak limit, as $N \to \infty$.
   $\mathcal{A}$ is defined by:

   $$\mathcal{A} = \left(\Omega, \Sigma, P_\lambda\right).$$

   Thus, $\mathcal{A}$ is a probability space.
3. The probability measure P_λ is called **Gibbs measure**.
4. The infinite interaction model $(X, \mathcal{A})$ is also called the **thermodynamic limit** or **macroscopic limit** of the finite interaction models $\left(\Lambda_N, X, U_{\Lambda_N} \mathcal{A}_{\Lambda_N}\right)$ as $N \to \infty$.

Remark

The problem of existence of the probability measure P_λ as a limit of the probability measures P_N is studied in literature but not in a rigorous mathematical context; it needs further investigation.

5. Particular Cases of Interaction Model

Ising Model

Ising model is an important mathematical model of ferromagnetism in statistical mechanics. It can be described as a particular case of the interaction model:

- each configuration $\omega : \mathbb{Z}^\nu \to \{\pm 1\}$;
- $P_0(X_s = 1) = P_0(X_s = -1) = 0.5$;
- $r = 1$;
- $\Phi_{\{s\}}(\omega) = \lambda h_s X_s(\omega)$, where $h_s \left(s \in \mathbb{Z}^\nu\right)$ are real numbers characterizing influence of an external field;
- $\Phi_{\{s,t\}}(\omega) = \lambda J_{st} X_s(\omega) X_t(\omega)$ if $\| s - t \| = 1$; here J_{st} are real numbers characterizing interaction of points s and t;
- in other cases $\Phi_B = 0$.

According to Definition 3.3, $U_\Lambda = U' + U''$, where:

$$U'(\omega) = \sum_{t \in \Lambda} h_t X_t(\omega),$$
$$U''(\omega) = \sum_{s,t \in \Lambda} \lambda J_{st} X_s(\omega) X_t(\omega).$$

After the single modification (by U') the distribution of X_t is still independent and is given by:

$$P'\left(X_t = -1\right) = \frac{e^{-h_t}}{e^{-h_t} + e^{h_t}} = \frac{1}{1 + e^{2h_t}}; \quad P'\left(X_t = 1\right) = \frac{e^{2h_t}}{1 + e^{2h_t}}.$$

After the plural modification (by U'') we get the Ising model, that is the modification by U_Λ, as proven in Theorem 2.3. Modification in two steps can simplify computations.

Some authors study the Ising model with four point interaction; see, for example, (Yang *et al.*, 2017) and references in it. In that case we add to the above definition of Φ_B a clause for a 4-point set B:

$$\Phi_B(\omega) = \lambda J_B \prod_{t \in B} X_t(\omega),$$

where B consists of the vertices of a unit square or the vertices of a tetrahedron with three edges of length 1 and three other edges of length $\sqrt{2}$.

Potts Model

Standard Potts model can be described as a particular case of the interaction model:

- each configuration $\omega : \mathbb{Z}^\nu \to \{1, 2, ..., q\}$;
- $P_0\left(X_t = i\right) = \frac{1}{q}, i = 1, 2, ..., q$;
- $r = 1$;
- $\Phi_{\{s,t\}}(\omega) = \lambda J_{st} \delta\left(X_s(\omega), X_t(\omega)\right)$ if $\| s - t \| = 1$;
  here $\delta(x, y) = \begin{cases} 1 & if\ x = y, \\ 0 & if\ x \neq y. \end{cases}$
  In other cases $\Phi_B = 0$.

The Ising model is a particular case of the standard Potts model when $q = 2$ (it can be reduced to the Ising model by linear transformation $X_t \to 2X_t - 3$).

6. Discussion

In this study we develop a mathematically rigorous concept of interaction model for a physical system with many particles, which interact with an external force and with one another; a random field $X_t\,(t \in \mathbb{Z}^{\nu})$ models some property of the system such as electric charge, density etc. We introduce a finite model first and then define the thermodynamic limit of the finite models with Gibbs probability measure. Unlike most existing models, in our model the set of values of X_t can be unbounded, which provides more generality.

We study properties of the interaction model. In particular, we split Gibbs modification of the initial independent probability measure into two steps reflecting the influence of an external field on the first step and the interaction between particles on the second step. The first-step modification is mathematically simple and leaves the field X_t independent, therefore this construction simplifies mathematical computations.

Next we show that Ising and Potts models are particular cases of the interaction model. If we change the set of values of the random field X_t from $\mathbb{R}$ to $\mathbb{R}^n$, then the generalized interaction model will also include the n-vector model and its special cases for $n = 2$ (XY-model) and $n = 3$ (Heisenberg model). We are planning to research further mathematical properties and physical applications of the interaction model.

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Author's Contributions

Farida Kachapova and Ilias Kachapov: Contributed to the preparation, development and publication of the manuscript.

Ethics

This is a mathematical article; no ethical issues can arise after its publication.

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