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Coordinated Transformation for Fuzzy Autocatalytic Set of Fuzzy Graph Type-3

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Abstract: Fuzzy Autocatalytic Set (FACS) of fuzzy graph Type-3 was established in the effort to describe the underlying phenomena of a clinical waste incineration process. Further research could be explored to scrutinize this concept especially in looking at its mathematical structure. In this paper, we highlighted the mathematical structures of FACS particularly on the algebraic structures of its matrix and graph. New definitions and theorems were generated through this investigation which leads to the visualization of the concept in Euclidean space. This visualization was accomplished through the transformation of non coordinated FACS to coordinated FACS in space. Eventually, the performance of coordinated FACS is then compared to the non-coordinated FACS in respect to the clinical waste incineration process.

Keywords: Fuzzy Graph, Fuzzy Autocatalytic Set, Incineration Process, Coordinated FACS

Introduction

The emergence of fuzzy graph to autocatalytic sets has instigated a new concept named Fuzzy Autocatalytic Set (FACS). A clinical waste incineration process in Malacca was successfully modeled using this concept (Sabariah, 2005) whereby its theoretical foundation was first laid by Tahir *et al*. (2010a; 2010b).

Definition 1.1

Let where E is a set of edges such that $E\{e_1,e_2,\ldots,e_n\}$. The fuzzy head of denotes as $h(e_i)$ and the fuzzy tail $t(e_i)$ are function of e_i such that $h: E \rightarrow [0,1]$ and $t:E \rightarrow [0,1]$ for $e_i \in E$. Fuzzy edge connectivity is a tuple $(t(e_i), h(e_i))$ and the set of all fuzzy edge connectivity is denoted as $C = \{(t(e_i),h(e_i)):e_i \in \text{E}\}$. The membership value of fuzzy edge connectivity is denoted as $\mu(e_i) = \min\{t(e_i),h(e_i)\}$ (Fig. 1).

Naturally, the adjacency matrix of FACS is defined as follows:

Definition 1.2

The entries for adjacency matrix of FACS of fuzzy graph Type-3; $C_{F_{ij}}$, is:

$$C_{F_{ij}} = \begin{cases} 0, & if\ i = j\ and\ e_i \notin E \\ \mu(e_i), & if\ i \neq j \end{cases}$$

Several new results of FACS which linked to Perron-Frobenius (PF) Theorem have been discussed in previous studies (Tahir *et al*., 2010a). In these studies, FACS was shown to provide better explanations of the real phenomena of the incineration process than crisp graph. Yet there are great opportunities and avenues for further exploration of this concept.

The main idea behind the definition of FACS is the merger of fuzzy graph Type-3 to autocatalytic set (Tahir *et al*., 2010b). The formal definition of FACS is given below.

Definition 1.3

Fuzzy Autocatalytic Set (FACS) is a subgraph where each nodes has at least one incoming link with membership value $\mu(e_i) \in (0,1], \forall e_i \in E$.

The transformation of any FACS of fuzzy graph Type-3 with no loop to a square matrix is then given by the following theorem (Sumarni *et al*., 2008).

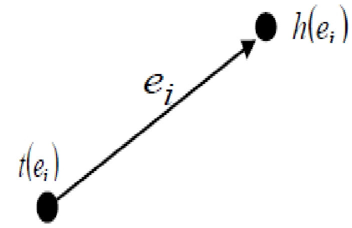

Fig. 1. Fuzzy head and tail of the i^{th} edge

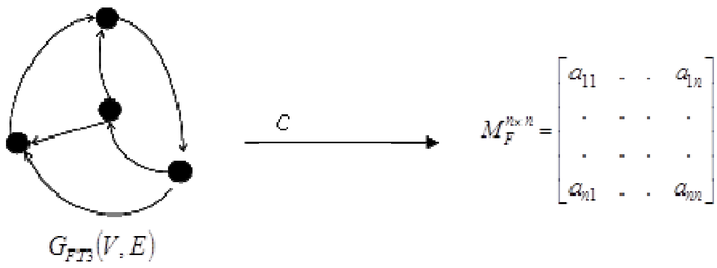

Fig. 2. $C: G_{FT3}(V, E) \rightarrow M_F^{n\times n}$

Theorem 1.1

Suppose $G_{\underset{FT3}{k}}(V,E)$ be a no loop fuzzy autocatalytic set of fuzzy graph of Type-3 where V is a set of vertices and E is a set of edges of the graph.

Defined that:

$$G_{\underset{FT3}{k}} = \begin{cases} 0 & when\ i - j\ and\ e_i \notin E \\ \mu(e_i) \in (0,1] & when\ i \neq j\ and\ e_i \in E \end{cases}\ for\ k = 1,2,\ldots,n$$

Let $G_{FT3} = \left\{ G_{\underset{FT3}{k}}; k = 1,2,\ldots,n \right\}$ be the finite set of all fuzzy graphs of Type-3 and let $M_F^{n\times n} = \left\{ [a_{ij}]_{n\times n} : a_{ij} \in [0,1]\ with\ a_{ii} = 0 \right\}$. Define $\theta: G_{FT3} \rightarrow M_F^{n\times n} \ni \theta(G_{iFT3}) = [a_{ij}]$ where $\mu(v_j, v_i) = a_{ij}$. Then $\theta: G_{FT3} \rightarrow M_F^{n\times n}$ is a bijective function.

The above theorem allows FACS of fuzzy graph Type-3 to be mapped to a square matrix as shown in Fig. 2.

Through this mapping, $M_F^{n\times n}$ was further deduced to Perron-Frobenious's features (Sumarni *et al*., 2008; 2009).

Materials and Methods

Some important features of FACS of fuzzy graph Type-3 can be explored through its corresponding adjacency matrix. Tahir *et al*. (2010b) have previously shown that if G is an FACS, the corresponding adjacency matrix contains a cycle and its Perron eigenvalue $\lambda_1 > 0$. Subsequently, further exploration of non negative matrix with zero diagonal reveals additional features of the matrix which includes irreduciblity. These new features are presented in the following lemma and theorems.

Lemma 2.1

Adjacency matrix deduces from FACS of fuzzy graph Type-3 has a similar matrix.

Proof

Let $B \in M_n$ be the FACS of fuzzy graph Type-3 such that:

$$B = \begin{cases} \mu_{ij} \in (0,1] & for(i,j) \in E\ and\ i \neq j \\ 0 & for(i,j) \notin E\ and\ i = j \end{cases}$$

Since $B \in M_n(R)$, the Real Schur Triangulation theorem (Horn and Johnson, 1985) guarantees that $\exists P$ which is orthogonal ($P^TP = P^T P = I$) such that $B = P^TAP$ and $A \in M_n(R)$. By taking $P^T = P^{-1}$, therefore $B = P^{-1}AP$. Hence, B is similar to A.

Theorem 2.2

The adjacency matrix $B = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$ such that $b_{\alpha\beta} \neq 0$ for $\alpha \neq \beta$ and $\alpha,\beta = 1,2,\ldots, n$ of FACS fuzzy graph Type-3 is primitive.

Proof

Let B be an adjacency matrix of FACS of fuzzy graph Type-3. Each b_{ii} since the graph is with no loop. Thus B is a zero diagonal matrix.

Since B is similar by Lemma 2.1, then there ∃ matrix P, A and P^{-1}, ∋ $B = P^{-1}AP$, where A is a diagonal matrix. Furthermore and A^k is also a diagonal matrix:

$$B^k = \underbrace{\left(P^{-1}AP\right)\left(P^{-1}AP\right)\ldots\left(P^{-1}AP\right)}_{k-times} = P^{-1}A^kP$$

By contradiction, Suppose $\nexists k \in N \ni B^k > 0$, therefore $\nexists k \in N \ni P^{-1}A^kP > 0$, with is diagonal matrix.

It means that $P^{-1}A^kP \ngtr 0$ for all k. Consequently, $P^{-1}A^1P \ngtr 0$. This means it contradicts to similarity of B (i.e., $B = P^{-1}AP$). Therefore, B must be primitive.

Moreover, using theorem on irreducibility as given below.

Theorem 2.3 (Horn and Johnson, 1985)

Let $A \in M_n$ and suppose $A \geq 0$. Then $A$is irreducible if $(I+A)^{n-1}>0$.

The following result on FACS is attained.

Theorem 2.4

If matrix $B \in M_{n\times n}$ of FACS of fuzzy graph Type-3 is primitive, then it is irreducible.

Proof

Let $B \in M_{n\times n}$ be a matrix of FACS of fuzzy graph Type-3 as defined in Def. 1.2, therefore B is a nonnegative matrix such that $b_{ij} \geq 0$ for all ij. Since B is primitive, choose $k>0$ such that $B^k>0$. Let $D = (I+B)^{n-1}B$, therefore $D = B^n + C_{n-1}B^{n-1} + \ldots + C_1B$ Consequently, $D>0 \Rightarrow (I + B)^{n-1}B>0 \Rightarrow (I + B)^{n-1}>0$ since $B^k>0$ for some $k \in N$. Hence B is irreducible by Theorem 2.3.

Perez *et al*. (2001) described length of edge for an undirected, simple and connected graph $G(V,E)$ by introducing a positive function l to possess the ordered property (Tahir *et al*., 2010b). He deduced that if it is an undirected, simple and connected graph with vertex set $V = \{v_1,v_2,\ldots,v_n\}$ and edge set $E = \{e_{ij}, e_{pq},\ldots e_{mn}\}$, then for every edge $e_{ij} = (v_i,v_j) \in E$, there exist a positive number $l(e_{ij}) = l((v_i,v_j))$ that represents the length of the edge.

The idea of Perez *et al*. (2001) as described above has initiated the following definition to represent length of an edge of FACS.

Definition 2.1

Let $G_{FT3}(V,E)$ be a no-loop Fuzzy Autocatalytic set of fuzzy graph Type-3 and $\theta_{ij} = \mu(e_{ij}) = \mu((v_i,v_j)) \in (0,1]$. A function $l{:}E \rightarrow R^+$ such that $l(\theta_{ij}) = \theta_{ij} \in R^+$ represents the length of the edge, (v_i,v_j).

Definition 2.1 leads to the following definition on length for directed path of FACS. This definition is necessary to establish a concept of aperiodicity of FACS.

Definition 2.2

Suppose $G_{FT3}(V,E)$ is a no-loop Fuzzy Autocatalytic set of fuzzy graph Type-3. The length of directed path that start and end at v_i with respect to r route is denoted as $K_r^{(i)}$ and $K_r^{(i)} = \max\{\theta_{ij},\theta_{st},\ldots,\theta_{qp},\theta_{pi}\}\forall r$.

The condition of aperiodicity is natural in the study for FACS of fuzzy graph Type-3 (Ninove, 2008). This is due to the fact that a particular route is different to the other route in the graph by its length or by vertices that it passes through. Precisely, aperiodicity denotes that a route is not periodic; which means that each route is not a repetition of the other route and thus length of each route in the graph is unique. This motivates the formation of the following definition which inspires from Euclid's algorithm for any two real numbers.

Definition 2.3

Let $K_r^{(i)}$ and $K_s^{(i)}$ be lengths of two routes of FACS atv_i, then the greatest common divisor (gcd) between these two routes is defined as:

$$\gcd\left(K_r^{(i)},K_s^{(i)}\right) = \begin{cases} 1 & if \nexists g,k,m \in Z \ni gk = K_r^{(i)}\ and\ gm = K_s^{(i)} \\ g & if \exists g,k,m \in Z \ni gk = K_r^{(i)}\ and\ gm = K_s^{(i)} \end{cases}$$

Definition 2.3 is a gateway to a permissible adoption and adaptation of some of the results in theordinary concept of greatest common divisor for integers.

Theorem 2.5

If $K_r^{(i)}$ and $K_s^{(i)}$ are lengths of two directed path of FACS at v_i, then $\gcd\left(K_r^{(i)},K_s^{(i)}\right) = 1$.

Proof

Let $\gcd\left(K_r^{(i)},K_s^{(i)}\right) = d$, therefore there exist integer a and b such that $K_r^{(i)} = da$ and $K_s^{(i)} = db$.

But then $da \leq 1$ and $db \leq 1$ by Definitions 2.1, 2.2 and $d \in Z^+$. If that the case $d \nless 1$, hence$d = 1$.

Corollary 2.6

If $K_r^{(i)}$ and $K_s^{(i)}$ are lengths of two directed path of FACS for $i \neq j$, then $\gcd\left(K_r^{(i)},K_s^{(j)}\right) = 1$.

Proof

Let $\gcd\left(K_r^{(i)}, K_s^{(j)}\right) = d$ for $i \neq j$. Therefore using similar argument as in Theorem 2.5, there existinteger *a* and *b* such that $K_r^{(i)}$ and $K_s^{(j)} = db$. But then $da \leq 1$ and and $d \in Z^+$. If that the case $d \nless 1$, hence $d = 1$.

Corollary 2.7:

$$\gcd\left(K_r^{(i)}, K_s^{(j)}, K_s^{(m)}\right) = 1$$

Proof

Let $K_r^{(i)}, K_s^{(j)}, K_s^{(m)}$ be lengths of three directed path of FACS at v_i, v_j and v_m respectively. But, $K_r^{(i)}, K_s^{(j)}, K_s^{(m)} \in (0,1]$ and $K_s^{(j)}, K_s^{(m)} \in (0,1]$. By Corollary 2.6, $\gcd\left(K_r^{(i)}, K_s^{(j)}, K_s^{(m)}\right) = 1$.

Theorem 2.8:

$$\gcd\left(K_1^{(i)}, K_2^{(i)}, K_3^{(i)}, ..., K_{n-1}^{(i)}, K_n^{(i)}\right)$$
$$= \gcd\left(K_1^{(i)}, K_2^{(i)}, K_3^{(i)}, ..., \left(K_{n-1}^{(i)}, K_n^{(i)}\right)\right)$$

Proof

Assume $\gcd\left(K_1^{(i)}, K_2^{(i)}, K_3^{(i)}, ..., K_{n-1}^{(i)}, K_n^{(i)}\right) = d_1$, $\gcd\left(K_1^{(i)}, K_2^{(i)}, K_3^{(i)}, ..., \left(K_{n-1}^{(i)}, K_n^{(i)}\right)\right) = d_2$ and $K_{n-1}^{(i)}, K_n^{(i)} = \gcd\left(K_{n-1}^{(i)}, K_n^{(i)}\right) = d_3$.

If $\gcd\left(K_1^{(i)}, K_2^{(i)}, K_3^{(i)}, ..., K_{n-1}^{(i)}, K_n^{(i)}\right) = d_1$, therefore $d_1 \left| K_j^{(i)} \right.$ for j = 1,..., n and consequently $d_1 \left| rK_{n-1}^{(i)} + sK_n^{(i)} = d_3 \right.$ for some *r*, $s \in Z$. On the other hand, $d_2 \left| K_j^{(i)} \right.$ for j = 1,...,n-2 and $d_2 | d_3$.

If that the case, $d_2 \left| K_{n-1}^{(i)} \right.$ and $d_2 \left| K_n^{(i)} \right.$ by transitivity of a divisor.

Hence, $d_1 = d_2$.

The definition of aperiodicity given by (Ninove, 2008) is as follows.

Definition 2.4

A strongly connected graph is aperiodic if, for every node $i \in N$, the greatest common divisor of the length of all paths from *i* to *i* is equal to one.

Consequently, the above results instigated the following theorem on the aperiodicity of FACS of fuzzygraph Type-3.

Theorem 2.9

Any graph of FACS of fuzzy graph Type-3 is aperiodic.

Proof

Let $G_{FT3}(V,E)$ be a graph of FACS. By Theorem 2.4, the adjacency matrix of FACS is irreduciblewhich imply that the graph is strongly connected. By using Definition 2.4 and Theorem 2.5, it isshown that any FACS of fuzzy graph Type-3 is aperiodic.

Similarly, this investigation on the algebraic structure of adjacency matrix can be extended to itstransition matrix.

Naturally, the formulation for the transition matrix for FACS of fuzzy graph Type-3 is based on itsfuzzy membership value (Sumarni *et al*., 2010).

Definition 2.5

The entries for transition matrix of FACS of fuzzy graph Type-3 is P_{ij}^* such that:

$$P_{ij}^* = \begin{cases} \dfrac{\theta_{ij}}{d_{out}(i)} & if \quad (i,j) \in E \\ 0 & otherwise \end{cases}$$

And the out-degree of i is with $d_{out}(i) = \sum \theta_{it} \ with \ \theta_{it} \in R^+$.

The definition for stochastic matrix (Stewart, 1994; Norris, 1998) is as follows.

Definition 2.6

A non-negative matrix is stochastic matrix if row sums equal to one.

FACS of fuzzy graph Type-3 can be shown to be stochastic by the following theorem.

Theorem 2.10

Transition matrix for no loop FACS of fuzzy graph Type-3 is stochastic.

Proof

Let $G_{FT3}(V,E)$ be a graph of FACS of fuzzy graph Type-3. Using Definition 2.5 of transition matrix of FACS and $P_{ii} = 0$ due to the non-existence of loop in the graph, therefore:

$$\sum_{j=1}^{n} P^*(i,j) = \sum_{j=1}^{n} \frac{\theta}{d_{out}(i)}$$
$$= \frac{1}{d_{out}(i)} \sum_{j=1}^{n} \theta_{ij}$$
$$= \frac{1}{d_{out}(i)} \left(d_{out}(i)\right) since \, d_{out}(i) = \sum_j \theta_{ij}$$
$$= 1$$

Hence P^* is stochastic.

The resultant matrix is also irreducible and is presented in the following theorem.

Theorem 2.11

Stochastic matrix for no loop FACS of fuzzy graph Type-3 is irreducible.

Proof

Let $P^* \in M_n$ (R) be a transition matrix for FACS of fuzzy graph Type-3 such that $P_{ij}^* = \begin{cases} \dfrac{\theta_{ij}}{d_{out}(i)} & if \quad (i,j) \in E \\ 0 & otherwise \end{cases}$ where the out-degree of i is $d_{out}(i) = \sum \theta_{it} \, with \theta_{it} \in R^+$.

Since $P \in M_n(R)$, Real Schur Triangulation theorem (Horn and Johnson, 1985) guarantees that $\exists P$ which is orthogonal ($P^TP= PP^T = I$) such that $P^* = P^TAP$. By taking $P^T = P^{-1}$, therefore $P^* = P^{-1}AP$. Hence, P^* is similar to A where A is a diagonal matrix. Furthermore, $(P^*)^k = \underbrace{(P^{-1}AP)(P^{-1}AP)...(P^{-1}AP)}_{k-time} = P^{-1}A^kP$ and k A is also a diagonal matrix. By contradiction, Suppose $\nexists k \in N \ni (P^*)^k > 0$, therefore $\nexists k \in N \ni P^{-1}A^kP > 0$, with A^k is diagonal matrix.

It means that $P^{-1}A^kP \not> 0$ for all k. Consequently, $P^{-1}A^1P \not> 0$. This means it contradicts to similarity of P^*(i.e., P^{-1} AP). Therefore, P^* must be primitive. Next, choose k>0 such that $(P^*)^k$>0 and let $D = (I + P^*)^{-1} P^*$, therefore $D = (P^*)^n + C_{n-1}(P^*)^{n-1} + ... + C_1(P^*)$. Consequently, $D > 0 \Rightarrow (I + (P^*))^{n-1}(P^*) > 0 \Rightarrow (I + (P^*))^{n-1} > 0$ since B^k>0 for some $k \in N$. Hence P^* is irreducible by Theorem 2.3 (Horn and Johnson, 1985).

The eigenvalues of Laplacian are related to structural properties of graph (Mohar, 1991). Chung defined thedirected Laplacian for aperiodic strongly connected directed graph by using transition probability matrix $P_{u,v}$ which is the probability of moving from vertex u to v as $P_{u,v} = \dfrac{w(u,v)}{d_{out}(u)}$ such that $d_{out}(u) = \sum_t w(u,t)$ is the out-degree of u and $w(u,v)$ is the weight of edge from u to v (Chung, 2005). However, in this work, edges of FACS are constructed based on fuzzy edge connectivity as stated in Definition 1.1. This situation leads to the formulation of the following new definition for directed Laplacian ofFACS.

Definition 2.7

Suppose P^* is the transition matrix for no loop FACS of fuzzy graph Type-3 with $P^*_{u,v}$ represents fuzzy value of moving from u to v. Let 1denotes the all 1s vector such that P^* 1 = 1. If the graph is strongly connected and a periodic, then PF Theorem guarantees that there exist a unique (row) vector, namely φ for which $\varphi P^* = \varphi$ with $\varphi(v)$>0 for all v and $\sum \varphi(v) = 1$. Let Φ be the diagonal matrix with $\Phi(v,v) = \Phi(v)$ then directed Laplacian of FACS is:

$$L = I - \frac{1}{2}\left(\Phi^{1/2}P^*\Phi^{-1/2}\left(P^*\right)^t\Phi^{1/2}\right)$$

where, by $(P^*)^t$ denotes the transpose of P^*. The combinatorial Laplacian of FACS is defined as:

$$L = \Phi - \frac{1}{2}\left(\Phi P^* + \left(P^*\right)^t \Phi\right)$$

Some of the properties for the directed Laplacian of FACS have been formulated in the previous study (Tahir *et al.*, 2010a; 2010b) in which they are summarized in the following theorem.

Theorem 2.12

Suppose $G_{FT3}(V,E)$ be a no loop of FACS of fuzzy graph of Type-3 which is strongly connected, then the combinatorial Laplacian $n \times n$ matrix L has the following properties:

- The matrix is real and symmetric
- The matrix is Hermitian
- The matrix is positive semi-definite and have n non-negative real eigenvalues which can beordered sequentially in an ascending order as $0 = \lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_{n-1}$.
- The smallest eigenvalue, $\lambda_0 = 0$ has multiplicity 1 where it is equal to the number of connected components in the graph

The procedure to transform the graph of FACS into 2D-Euclidean space is adopted from Carmel *et al.* (2002) which is based on Laplacian matrix and solving a unique one dimensional optimization problem in order to determine their coordinates. The general overview of the transformation is depicted in the following Fig. 3.

In order to transform the graph of FACS to coordinated graph, definition of balanced vector and optimal arrangement of directed graph (Carmel *et al.*, 2002) is adopted. Thus, balanced vector and optimal arrangement with regard to FACS can be presented as follows.

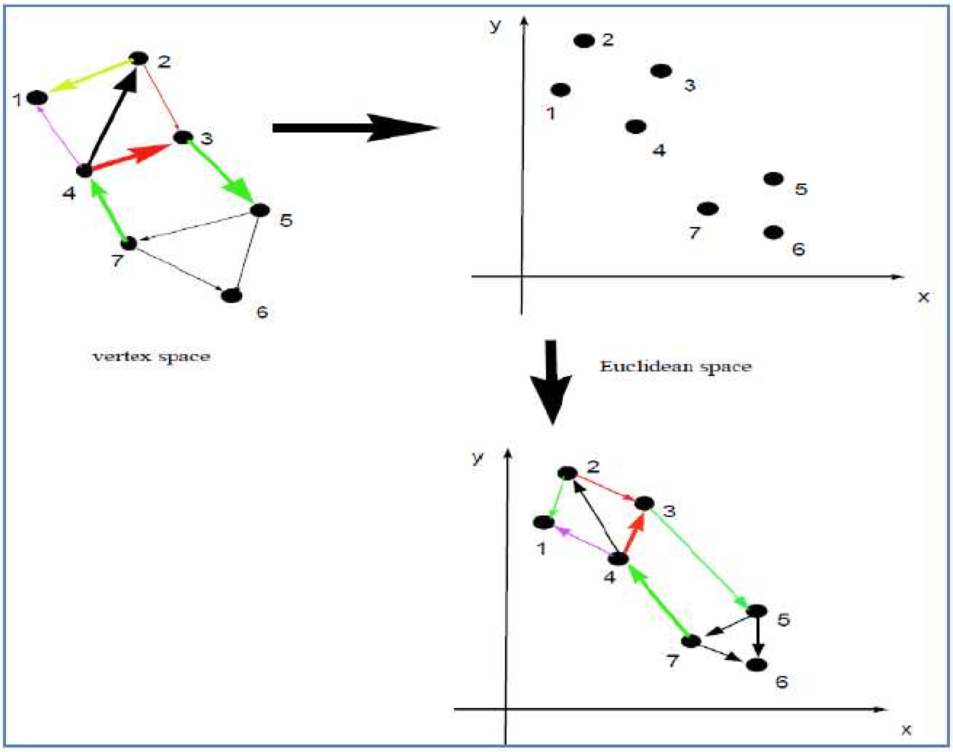

Fig. 3. Schematic illustration transformation of FACS from vertex space to Euclidean space

Definition 2.8

Let G_{FT3} (V,W^*) be a graph of FACS where W^* is $n\times n$ matrix of edge weight for FACS, i.e., $w_{ij} = \theta_{ij}$. The balance of the i^{th} node is a vector $b = (b_1,\ldots,b_n)^t$ such that $bi = \sum_{j=1}^{b}\theta_{ij} - \sum_{j=1}^{b}\theta_{ji}$.

Consequently, $y = (y_1,\ldots,y_n)$ is determined by using concept of minimization of hierarchy energy function where it is equivalent to solving an optimal arrangement (Carmel *et al*., 2002). Similarly, the same conceptcan be incorporated to FACS and it is redefined as follows.

Definition 2.9

Let $G_{FT3}(V,W^*)$ be a graph of FACS with Laplacian, L and balance, b. Its optimal arrangement, y^*, is the solution for $Ly = b$, subject to the constraint $y^t.1_n = 0$.

The optimal arrangement y^* is the vector for y-coordinates of FACS and is solved by using Conjugate Gradient method.

On the other hand, $x = (x_1,\ldots,x_n)^t$ representing the x-coordinates of FACS is solved using minimization of edge squared lengths via Tutte-Hall energy function, $E = \frac{1}{2}\sum_{i,j=1}^{n} w_{ij}\left(x_i - x_j\right)^2 = x^t Lx$.

According to (Carmel *et al*., 2002), minimization of this energy function is the Fiedler vector, which is the eigenvector of the Laplacian associated with the smallest positive eigenvalue. In this case, the Fiedler vector of combinatorial directed Laplacian is obtained by solving its eigenvalue problem.

From the properties deduced for combinatorial Laplacian matrix of FACS of fuzzy graph Type-3 and method of finding the (x, y) coordinate of FACS as described earlier, the procedure for the transformation can be presented formally by the following theorem.

Theorem 2.13

Any FACS of fuzzy graph of Type-3 can be induced to coordinated FACS of fuzzy graph of Type-3.

Proof

Suppose G_{FT3} (V,E) is an FACS of fuzzy graph Type-3 with n vertices and fulfill all the conditions described in Theorem 2.1, Theorem 3.3, Theorem 3.5 and Corollary 3.6. Next, define transition matrix for FACS of fuzzy graph type-3, P^*as in Definition 4.1 which fulfill all the conditions described in Theorem 4.1 and Theorem 4.2. Next, find left Perron vector, φ, for P^* using standard matrix multiplication $(P^*)2^{n-1} = (P^*)^n\ (P^*)^{n-1}$ or by using Theorem 1 in (Anton and Rorres, 1977). Next, construct combinatorial Laplacian for FACS, L, as defined in Definition 2.7 and is satisfying Theorem 2.12. Let balance vector for

$G_{FT3}(V,E)$, is b, as in Definition 2.8 and define optimal arrangement, y^* as in Definition 2.9. Then$y$-coordinates of the nodes of G_{FT3} (V,E) is given by y^*. Herewith, solve eigenvalue problem of L toget the Fiedler vector, $x = (x_1,\ldots,x_n)^T$ which represents x-coordinates of the nodes. Positioned the nodes in 2D-Euclidean space using (x, y) coordinate obtained earlier and draw its corresponding edges. Finally FACS of fuzzy graph Type-3 is induced to coordinated FACS of fuzzy graph Type-3.

The following corollaries are immediate cases of Theorem 2.13.

Corollary 2.14

FACS of fuzzy graph Type-3 described in Sabariah's (2005) can be induced to coordinated FACS offuzzy graph Type-3.

Proof

From Theorem 5.2 of (Tahir *et al*., 2010a), the graph for clinical waste incineration process in (Sabariah, 2005) can be represented as $G_{d_f} = \left(V, E\left(\mu\left(e_i\right)\right)\right)$ for i = 1,...,15 where $\mu(e_i) = \min\{t(e_i), h(e_i)\}$ which is FACS of fuzzy graph Type-3. By using Theorem 2.10 above, immediately it can be induced to coordinated FACS of fuzzy graph Type 3.

Now by letting $i{:}V\rightarrow V_F$ the following corollary can be obtained.

Corollary 2.15

FACS of fuzzy graph with fuzzy vertices and edges can be induced to coordinated fuzzy graph.

Proof

Consider G_{FT3} (V_F, E_F), then each vertices in V_F has fuzzy membership value; i.e., $(V_i, \mu(V_i))\forall i$. By replacing V_i with $(V_i,\mu(V_i))$ in Theorem 2.10, any FACS with fuzzy vertices and edges can thereforebe induced to coordinated graph.

Results and Discussion

A clinical waste incineration process was modeled by FACS of fuzzy graph Type-3 (Sabariah, 2005; Tahir *et al*., 2010b) as in Fig. 4.

In this present work, FACS representing the clinical waste incineration process can be transformed into 2D-Euclidean space stated in Collorary 2.14 by using Theorem 2.13 as in Fig. 5.

Every edge of the graph in Fig. 4 is associated to a membership value for fuzzy edge connectivity whereby no information on location of each node in this graph. In contrast, every edge of the graph in Fig. 5 provides not only information on membership value, it also gives the information of its length. Here, every node has its own coordinate and the location of every node is given in Table 1.

Table 1. Location of variable (node) of FACS of fuzzy graph of Type-3 of an incineration process

x_i	Variable	x-coordinates	y-coordinates $\times 10^{-5}$
1	waste	0.083174265533019	0.8110
2	fuel	-0.843473193361576	1.9506
3	oxygen	0.083185051005536	0.8108
4	carbon dioxide	0.083176199384581	0.8111
5	carbon monoxide	0.510763130563702	1.2933
6	other gases including water	0.083174546799956	0.8110

Table 2. Euclidean distance of the edges in FACS

	V_1 (waste)	V_2 (fuel)	V_3 (O_2)	V_4 (CO_2)	V_5 (CO)	V_6(H_2O &otp*)
V_1(waste)	0	2.7616×10^5	20.0	10.00	2.1043×10^5	2.8127×10^{-7}
V_2(fuel)	2.7616×10^5	0	2.7614×10^5	2.7617×10^5	6.5730×10^4	1.9506×10^5
V_3(O_2)	20.0	2.7614×10^5	0	30.00	2.1041×10^5	20.0
V_4(CO_2)	10.00	2.7617×10^5	30.00	0	2.1044×10^5	10.00
V_5(CO)	2.1043×10^5	6.5730×10^4	2.1041×10^5	2.1044×10^5	0	2.1043×10^5
V_6(H_2O &otp*)	2.8127×10^{-7}	1.9506×10^5	20.0	10.00	2.1043×10^5	0
Average distance	4.8661×10^5	1.0893×10^6	4.8662×10^5	4.8666×10^5	9.0744×10^5	4.0552×10^5

Table 3. Results of graph dynamic procedure for non-coordinated and coordinated FACS

Non-coordinated FACS	Coordinated FACS	
Sequence of depleted variables	Fuel, CO, O_2, waste	Fuel, CO, O_2, CO_2
The most important variables in the incineration process	-	Waste and H_2O and other pollutants
By-product of the incineration process	CO_2, H_2O and other pollutants	H_2O and other pollutants

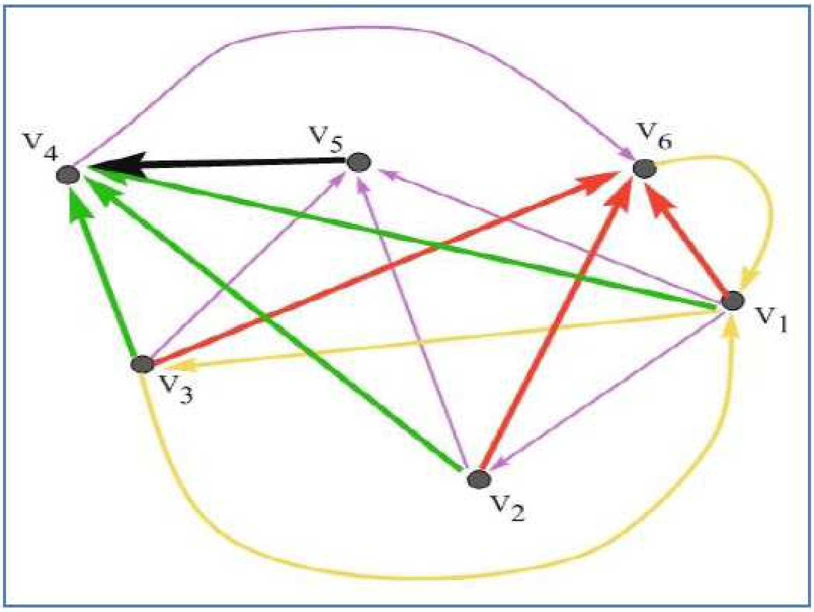

Fig. 4. FACS of fuzzy graph Type-3 for the clinical waste incineration process

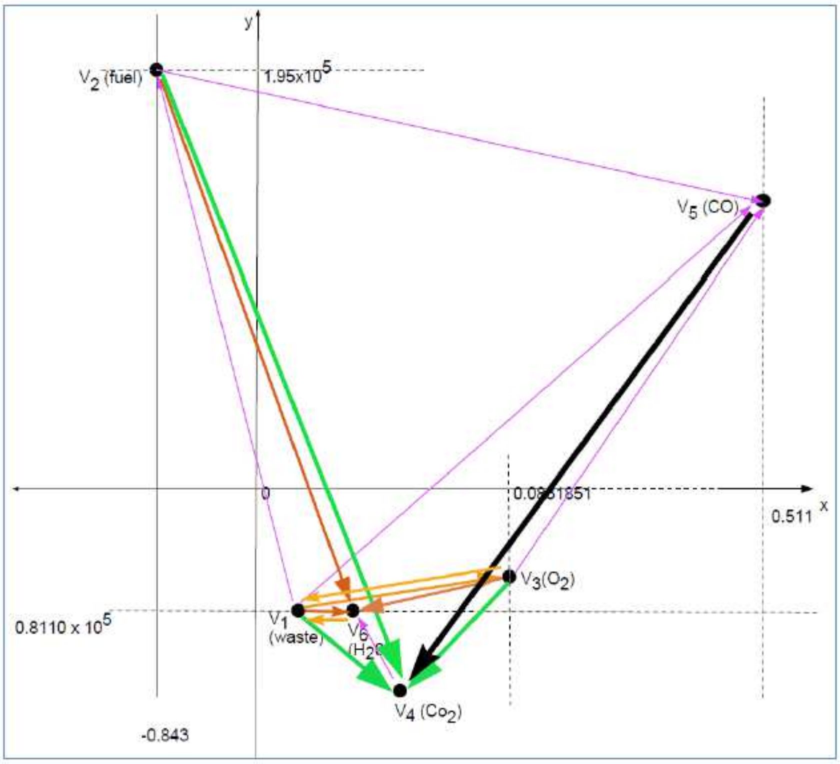

Fig. 5. FACS of fuzzy graph Type-3 of the clinical waste incineration process in 2D-Euclidean space

From the coordinates given in Table 1, the edge's length between two nodes is calculated as shown in Table 2.

The graph dynamic procedure for non coordinated FACS of fuzzy graph Type-3 was discussed in (Tahir *et al*., 2010b). The work was extended to coordinated FACS and the comparison between the two is given in Table 3.

Table 3 shows that coordinated FACS provides not only the information on the sequence of depleted variables and the information on the by-product, but it also gives the set of most important variables in the incineration process.

Conclusion

Mathematical structures of FACS of fuzzy graph Type-3 has been investigated through its adjacency matrix, transition matrix and Directed Laplacian matrix. This mathematical structure provides a platform to transform FACS to coordinated FACS. Comparison between the results of graph dynamic procedure for non-coordinated and coordinated FACS has shown that coordinated FACS is superior to non-coordinated FACS with respect to information for sequence of depleted, important and detail of by-product of variables in the clinical waste incineration process.

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Author's Contributions

Tahir Ahmad: Designed the research plan and organized the study. Coordinated the overall framework, contributed to the writing and reviewing the manuscript critically.

Sumarni Abu Bakar: Designed the research plan. Contributed in giving ideas, writing and proofreading the manuscript.

Sabariah Baharun: Participated in reviewing the manuscript and giving ideas.

Faisal A.M. Binjadhnan: Participated in giving ideas and reviewing the manuscript.

Ethics

This article is original and contains unpublished material. The correspondingauthorconfirms that all of the authors have read and approved the manuscript and no ethical issues involved.

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