Generalization of (0, 4) Lacunary Interpolation by Quantic Spline

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Abstract: Problem statement: Spline functions are the best tool of polynomials used as the basic means of approximation theory in nearly all areas of numerical analysis. Also in the problem of interpolation by g-spline construction of spline, existences, uniqueness and error bounds needed. **Approach:** In this study, we generalized (0,4) lacunary interpolation by quanta spline function. **Results:** The results obtained, the existence uniqueness and error bounds for generalize (0, 4) lacunary interpolation by qunatic spline. **Conclusion:** These generalize are preferable to interpolation by quantic spline to the use (0,4).

Key words: Spline function, existence and uniqueness, error bounds

INTRODUCTION

Spline functions are the best tool of polynomials used as the basic means of approximation theory in nearly all areas of numerical analysis. One uses polynomial for approximation because they can be evaluated, differentiated and integrated easily and in finitely many steps using the basic arithmetic operations of addition, subtraction, division and multiplication. Spline functions constitute a relativity new subject in analysis. During the past twentieth both the theories of splines and experiences with their use in numerical analysis have under gone a considerable degree of development. The following works deal to various degree with the theory and application of splines, (Ahlberg *et al*., 1967). In addition to the papers mentioned above dealing with best interpolation or approximation by splines, there were also a few papers that deal with constructive properties of space of spline interpolation (Kanth *et al*., 2006; Khan and Aziz, 2003; Siddiqi *et al*., 2007). In this study we studied the generalization of one type of lacunary interpolation by quautic spline this type is (0,4) but in works (Varma, 1978; Venturino, 1996) showed this type but not in general. Also in the future we can use the same idea for different lacunary interpolation that means we can generalities for different cases in the subjected lacunary interpolation by spline.

We have Hermite interpolation if for each i, the order j of derivatives in (1) from unbroken Sequence. If some of the sequences are broken, we have lacunary interpolation.

The lacunary interpolation problem, which we have investigated in this study, consists in finding the five degree spline $S(x)$, interpolating data given on the function value and fourth order in the interval [0,1]. Also, an extra initial condition is prescribed on the first derivative.

This study is organized as follows: First consider the spline function of degree five is presented which interpolates the lacunary data (0,4). Some theoretical results about existence, uniqueness and error bounds of the spline function of degree five are introduced and also convergence analysis is studied. To demonstrate the convergence of the prescribed lacunary spline function.

MATERIALS AND METHODS

Descriptions of the method: We present for the first time according to our knowledge a five degree spline (0,4) interpolation for one dimensional and given sufficiently smooth function $f(x)$ defined on $i = [0,1]$ and:

$$p_n^{(j)}(x_i) = a_{i,j}, \qquad i = 1, 2, ..., n; j = 0, 1, 2, ..., n \qquad (1)$$

We have Hermite interpolation if for each i, the order j of derivatives in (1) from unbroken Sequence. If some of the sequences are broken, we have lacunary interpolation:

$$\Delta_n : 0 = x_0 < x_2 < \ldots x_n = 1$$

Denote the uniform partition of i with knots:

$$x_i = ih$$

Where:
h = $x_{i+1} - x_i$
i = 0,1,...,n-1

We define the class of spline function $S_{n,5}^{(2)}$ where $S_{n,5}^{(2)}$ denotes the class of all splines of degree five which belongs to $C^2[0,1]$ and n is the number of knots, as follow:

Any element $S_\Delta(x) \in S_{n,5}^{(2)}$ if the following two conditions are satisfied:

$$\left. \begin{array}{l} \text{(i)} \ S_\Delta(x) \in C^2[0,1] \\ \text{(ii)} \ S_\Delta(x) \text{ is aplonomial of degree five in} \\ \quad \text{each } [x_i, x_{i+1}], i = 0,1,...,n-1 \end{array} \right\} \quad (2)$$

Construction of the spline function: If P(x) is a polynomial of degree five on [0,1], then we have:

$$\begin{array}{l} P(x) = P(0)A_0(x) + P(\lambda)A_1(x) + P(1)A_2(x) + \\ \quad P'(0)A_3 P'(1)A_4(x) + P^{(4)}(\lambda)A_5(x) \end{array} \quad (3)$$

$\forall \lambda \in [0,1] / \left\{ 0, \frac{1}{2}, 1 \right\}$, Where:

$$\left. \begin{array}{l} A_0(x) = \dfrac{1}{2\lambda^2(2\lambda-1)} \left(\begin{array}{l} (2\lambda+1)x^5 - 5\lambda(2\lambda+1)x^4 + \\ (8\lambda^3 + 16\lambda^2 + 4\lambda - 3)x^3 - (12\lambda^3 + \\ 4\lambda^2 + \lambda - 2)x^2 + 2\lambda^2(2\lambda-1) \end{array} \right) \\[4mm] A_1(x) = \dfrac{1}{2\lambda^2(\lambda-1)^2(2\lambda-1)} \left(-x^5 + 5\lambda x^4 - 10\lambda - 3)x^3 \right) \\[4mm] A_2(x) = \dfrac{-1}{2(\lambda-1)^2(2\lambda-1)} \left(\begin{array}{l} (2\lambda-3)x^5 - 5\lambda(2\lambda-3)x^4 + \\ (8\lambda^3 - 20\lambda+5)x^3 - \lambda(12\lambda^2 - \\ 20\lambda+5)x^2 \end{array} \right) \\[4mm] A_3(x) = \dfrac{1}{2\lambda(2\lambda-1)} \left(\begin{array}{l} x^5 - 5\lambda x^4 + (4\lambda^2 + 8\lambda - 3)x^3 - \\ (8\lambda^2 + \lambda - 2)x^2 + 2\lambda(2\lambda-1)x \end{array} \right) \\[4mm] A_4(x) = \dfrac{1}{2\lambda(2\lambda-1)} \left(\begin{array}{l} x^5 - 5\lambda x^4 + (4\lambda^2 + 4\lambda - 1)x^3 - \\ \lambda(4\lambda-1)x^2 \end{array} \right) \\[4mm] A_5(x) = \dfrac{1}{48(2\lambda-1)} \left(x^5 - (\lambda+2)x^4 + (2\lambda+1)x^3 - \lambda x^2 \right) \end{array} \right\} \quad (4)$$

In the subsequent section we need the following values: $f \in C^5[0,1]$ we have the following expansions:

$$\left. \begin{array}{l} f(x_{i-1}) = f(x_i) - hf'(x_i) + \dfrac{1}{2}h^2 f''(x_i) + \dfrac{1}{6}h^3 f'''(x_i) + \\[3mm] \quad \dfrac{1}{24}h^4 f^{(4)}(x_i) - \dfrac{1}{120}h^5 f^{(5)}(\theta_{1,i}), x_{i-1} < \theta_{1,i} < x_i \\[3mm] f(x_{i+1}) = f(x_i) - hf'(x_i) + \dfrac{1}{2}h^2 f''(x_i) - \dfrac{1}{6}h^3 f'''(x_i) + \\[3mm] \quad \dfrac{1}{24}h^4 f^{(4)}(x_i) - \dfrac{1}{120}h^5 f^{(5)}(\theta_{2,i}), x_i < \theta_{2,i} < x_{i+1} \\[3mm] f(x_{i-1+\lambda}) = f(x_i) - (\lambda-1)hf'(x_i) + \dfrac{(\lambda-1)^2}{2}h^2 f''(x_i) - \\[3mm] \quad \dfrac{(\lambda-1)^3}{6}h^3 f^{(3)}(x_i) + \dfrac{(\lambda-1)^4}{24}h^4 f^{(4)}(x_i) - \\[3mm] \quad \dfrac{(\lambda-1)^5}{120}h^5 f^{(5)}(\theta_{3,i}), x_i < \theta_{3,i} < x_{i-1+\lambda} \\[3mm] f(x_{i+\lambda}) = f(x_i) - \lambda hf'(x_i) + \dfrac{\lambda^2}{2}h^2 f''(x_i) - \\[3mm] \quad \dfrac{\lambda^3}{6}h^3 f^{(3)}(x_i) + \dfrac{\lambda^4}{24}h^4 f^{(4)}(x_i) + \\[3mm] \quad \dfrac{\lambda^5}{120}h^5 f^{(5)}(\theta_{4,i}), x_i < \theta_{4,i} < x_{i+\lambda} \\[3mm] f^{(4)}(x_{i-1+\lambda}) = f^{(4)}(x_i) + (\lambda-1)hf^{(5)}(\theta_{5,i}), x_{i-1+\lambda} < \theta_{5,i} < x_i \\[3mm] f^{(4)}(x_{i+\lambda}) = f^{(4)}(x_i) + \lambda hf^{(5)}(\theta_{6,i}), x_i < \theta_{6,i} < x_{i+\lambda} \\[3mm] f'(x_{i-1}) = f'(x_i) - hf''(x_i) + \dfrac{1}{2}h^2 f^{(3)}(x_i) - \dfrac{1}{6}h^3 f^{(4)}(x_i) + \\[3mm] \quad \dfrac{1}{24}h^4 f^{(5)}(\theta_{7,i}), x_{i-1} < \theta_{1,i} < x_i \\[3mm] f'(x_{i+1}) = f'(x_i) - hf''(x_i) + \dfrac{1}{2}h^2 f^{(3)}(x_i) + \dfrac{1}{6}h^3 f^{(4)}(x_i) + \\[3mm] \quad \dfrac{1}{24}h^4 f^{(5)}(\theta_{8,i}), x_i < \theta_{8,i} < x_{i+i} \\[3mm] f^{(3)}(x_{i+\lambda}) = f^{(3)}(x_i) - \lambda hf^{(4)}(x_i) + \dfrac{\lambda^2}{2}h^2 f^{(5)}(\theta_{9,i}), \\[3mm] \quad x_i < \theta_{9,i} < x_{i+\lambda} \end{array} \right\} \quad (5)$$

Main results: The existence and uniqueness theorem for spline function of degree five which interpolate the lacunary data (0, 4) are presented and examined.

Theorem 1: Existence and Uniqueness spline functions: Given arbitrary numbers $f(x_i)$, $f^{(r)}(x_{i+\lambda})$, i = 0,1,...,n-1; r = 0, 4 and $f''(x_0)$, $f'(x_n)$ there exists a unique spline $S_n(x) \in S_{n,5}^{(2)}$ such that:

$$\left. \begin{array}{l} S_n(x_i) = f(x_i), i = 0,1,...,n-1 \\ S_n^{(r)}(x_{i+\lambda}) = f^{(r)}(x_{i+\lambda}), i = 0,1,...,n-1; r = 0,4 \\ S_n''(x_0) = f''(x_0), S_n''(x_n) = f''(x_n) \end{array} \right\} \quad (6)$$

Proof: The proof depends on the following representation of $S_n(x)$ for $x_i \leq x \leq x_{i+1}$, $i = 0,1,\ldots n-1$ we have:

$$Sn(x) = f(x_i)A_0\left(\frac{x-x_i}{h}\right) + f(x_i + \lambda)A_1\left(\frac{x-x_i}{h}\right) +$$

$$f(x_{i+1})A_2\left(\frac{x-x_i}{h}\right) + hS_n'(x_i)A_3\left(\frac{x-x_i}{h}\right) + \qquad (7)$$

$$hS_n'(x_{i+1})A_4\left(\frac{x-x_i}{h}\right) + h^4 f^4(x_{i+\lambda})A_5\left(\frac{x-x_i}{h}\right)$$

On using Eq. 7 and the conditions:

$$S_n''(0) = f''(0), S_n''(1) = f''(1) \qquad (8)$$

We see that $S_n(x)$ as given by (7) satisfies (2) and is five tic in $[x_i, x_{i+1}]$, $i = 0, 1,\ldots,n-1$. We also need to show that whether it is possible to determine $S_n''(x_i), i = 1,2,\ldots,n-1$ uniquely. For this purpose we use the fact that:

$$S_n''(x_{i+}) = S_n''(x_{i-}), i = 1,2,\ldots,n-1$$

Where:

$$S_n''(x_{i+}) = \lim_{x \to x_i} + S_n''(x)$$

and:

$$S_n''(x_{i-}) = \lim_{x \to x_i} - S_n''(x)$$

with the help of (7) and (8) reduced to:

$$\lambda(\lambda-1)^3(4\lambda-3)hS_n'(x_{i-1}) + 4\lambda(\lambda-1)\left(4\lambda - \frac{1}{2}\right)$$

$$(4\lambda^2 - 4\lambda - 1)hS_n'(x_i) + \lambda^3(\lambda-1)(4\lambda-1)$$

$$hS_n'(x_{i+1}) = -(\lambda-1)^3(12\lambda^2 - 4\lambda - 3)f(x_{i-1}) -$$

$$(20\lambda^4 - 40\lambda^3 + 15\lambda^2 + 5\lambda - 2)f(x_i) + \lambda^3 \qquad (9)$$

$$(12\lambda^2 - 20\lambda + 5)f(x_{i+1}) - (5\lambda-3)f(x_{i-1+\lambda}) +$$

$$(5\lambda-2)f(x_{i+\lambda}) + \frac{1}{24}\lambda^2(\lambda-1)^3 - \frac{1}{24}\lambda^3(\lambda-1)^2 h^4 f^{(4)}(x_{i+\lambda})$$

$i = 1,2,\ldots,n-1$. Equation 9 is a strictly tri-diagonal dominant system which has a unique solution. Thus $S_n''(x_{2i}), i = 1,2,\ldots,n-1$ can be obtained uniquely by the system (9) which established Theorem 1.

Convergence and error bounds: The error bound of the spline function $S(x)$ which is a solution of the problem (6) is obtained for the uniform partition I by the following Lemma:

Lemma 1: Let us write $B_i = \left|S_n''(x_i) - f'(x_i)\right|$, then for $f \in C^5[0,1]$, we have:

$$\max B_i \leq$$

$$\frac{(10\lambda^5 + 12\lambda^4 - 18\lambda^3 + 12\lambda^2 + 2\lambda - 3)}{10\lambda\left(\lambda - \frac{1}{2}\right)(12\lambda^2 - 12\lambda + 1)} \frac{h^4}{24} w(f^{(5)};h) \qquad (10)$$

for $i = 1,2,\ldots,n-1$ Where:

$$w\left(f^{(5)}; \frac{1}{n}\right) = \max\left\{\left|f^{(5)}(x) - f^{(5)}(y)\right|; \left|x - y\right| \leq \frac{1}{n}\right\}$$

Proof: From (8) we have:

$$\lambda(\lambda-1)^3(4\lambda-3)h\left(S_n'(x_{i-1}) - f'(x_{i-1})\right) +$$

$$4\lambda(\lambda-1)\left(4\lambda - \frac{1}{2}\right)(4\lambda^2 - 4\lambda - 1)h\left(S_n'(x_i) - f(x_i)\right) +$$

$$\lambda^3(\lambda-1)(4\lambda-1)h\left(S_n'(x_{i+1}) - f(x_{i+1})\right) - (\lambda-1)^3$$

$$(12\lambda^2 - 4\lambda - 3) = -(\lambda-1)^3(12\lambda^2 - 4\lambda - 3)f(x_{i-1}) -$$

$$(20\lambda^4 - 40\lambda^3 + 15\lambda^2 + 5\lambda - 2)f(x_i) + \lambda^3$$

$$(12\lambda^2 - 20\lambda + 5)f(x_{i+1}) - (5\lambda-3)f(x_{i-1+\lambda}) +$$

$$(5\lambda-2)f(x_{i+\lambda}) + \frac{1}{24}\lambda^2(\lambda-1)^3 h^4 f^{(4)}(x_{i+\lambda}) -$$

$$\lambda(\lambda-1)^3(4\lambda-3)hf'(x_{i-1}) - 4\lambda(\lambda-1)\left(\lambda - \frac{1}{2}\right)$$

$$(4\lambda^2 - 4\lambda - 1)hf'(x_i) - \lambda^3(\lambda-1)(4\lambda-1)$$

$$hf'(x_{i+1}) = \frac{1}{120}(\lambda-1)^3(12\lambda^2 - 4\lambda - 3)h^5 f^{(5)}(\theta_{1,i}) +$$

$$\frac{1}{120}\lambda^3(12\lambda^2 - 20\lambda + 5)h^5 f^{(5)}(\theta_{2,i}) - \frac{1}{120}(\lambda-1)^5$$

$$(5\lambda-3)h^5 f^{(5)}(\theta_{3,i}) + \frac{1}{120}\lambda^5(5\lambda-1)h^5 f^{(5)}(\theta_{4,i}) +$$

$$\frac{1}{24}\lambda^2(\lambda-1)^4 h^5 f^{(5)}(\theta_{5,i}) - \frac{1}{24}\lambda^4(\lambda-1)^2 h^5 f^{(5)}(\theta_{6,i}) -$$

$$\frac{1}{24}\lambda(\lambda-1)^3(4\lambda-3)h^5 f^{(5)}(\theta_{7,i}) - \frac{1}{24}\lambda^3(\lambda-1)(4\lambda-1)$$

$$h^5 f^{(5)}(\theta_{8,i}) \leq \frac{h^5}{120}(\lambda-1)$$

$$(10\lambda^5 + 12\lambda^4 - 18\lambda^3 + 12\lambda^2 + 2\lambda - 3)h^2 \; \alpha_1 w\left(f^{(5)}; \frac{1}{n}\right)\left|\alpha_1\right| \leq 1$$

The result (10) follows on using the property of diagonal dominant.

Lemma 2: Let $f \in C^5[0,1]$ then:

$$\left| S_n^{(5)}(x_{i+}) - f^{(5)}(x_i) \right| \le$$
$$\frac{(120\lambda^6 - 220\lambda^5 + 274\lambda^4 - 166\lambda^3 + 34\lambda^2 + 4\lambda - 6)}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)(12\lambda^2 - 12\lambda + 1)} \quad (11)$$
$$w\left(f^{(5)}; \frac{1}{n}\right)$$

$$\left| S_n^{(5)}(x_{i+\lambda}) - f^{(5)}(x_i) \right| \le$$
$$\frac{(120\lambda^6 - 220\lambda^5 + 274\lambda^4 - 166\lambda^3 + 34\lambda^2 + 4\lambda - 6)}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)(12\lambda^2 - 12\lambda + 1)} \quad (12)$$
$$w\left(f^{(5)}; \frac{1}{n}\right)$$

$$\left| S_n^{(5)}(x_{i-}) - f^{(5)}(x_i) \right| \le$$
$$\frac{(120\lambda^7 - 400\lambda^6 + 656\lambda^5 - 673\lambda^4 + 372\lambda^3 - 88\lambda^2 - 3\lambda + 3)}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2 - 12\lambda + 1)} \quad (13)$$
$$w\left(f^{(5)}; \frac{1}{n}\right)$$

$$\left| S_n^{(3)}(x_{i+\lambda}) - f^{(3)}(x_{i+\lambda}) \right| \le$$
$$\frac{\left(\begin{array}{l}960\lambda^9 - 3000\lambda^8 + 3236\lambda^7 - 1504\lambda^6 - \\ 242\lambda^5 + 424\lambda^4 + 126\lambda^3 - 42\lambda^2 + 42\lambda + 9\end{array}\right)}{\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2 - 12\lambda + 1)} \frac{h^2}{160} w\left(f^{(5)}; \frac{1}{n}\right) \quad (14)$$

$$\left| S_n^{(2)}(x_{i+\lambda}) - f^{(2)}(x_{i+\lambda}) \right| \le$$
$$\frac{\left(\begin{array}{l}480\lambda^{10} - 1760\lambda^9 + 2072\lambda^8 - 308\lambda^7 - 1500\lambda^6 + \\ 1288\lambda^5 - 376\lambda^4 - 130\lambda^3 + 100\lambda^2 - 2\lambda - 6\end{array}\right)}{\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)(12\lambda^2 - 12\lambda + 1)} \frac{h^3}{240} \quad (15)$$
$$w\left(f^{(5)}; \frac{1}{n}\right)$$

and

$$\left| S_n^{'}(x_{i+\lambda}) - f'(x_{i+\lambda}) \right| \le$$
$$\frac{\left(\begin{array}{l}57600\lambda^{10} - 172860\lambda^9 + 174798\lambda^8 - \\ 26752\lambda^7 - 68324\lambda^6 + 46864\lambda^5 - \\ 10530\lambda^4 + 667\lambda^3 + 42\lambda^2 - 31\lambda + 6\end{array}\right)}{\lambda(\lambda-1)\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2 - 12\lambda + 1)} \frac{h^3}{240} w\left(f^{(5)}; \frac{1}{n}\right) \quad (16)$$

Proof: From (7) we have:

$$h^5 S_n^{(5)}(x_{i+}) = \frac{60(2\lambda-1)}{\lambda^2(2\lambda-1)} f(x_i) - \frac{60}{\lambda^2(\lambda-1)^2(2\lambda-1)} f(x_{i+\lambda}) -$$
$$\frac{60(2\lambda-3)}{(\lambda-1)^2(2\lambda-1)} f(x_{i+1}) + \frac{60}{\lambda(2\lambda-1)} h S_n^{'}(x_i) +$$
$$\frac{60}{(\lambda-1)(2\lambda-1)} h S_n^{'}(x_{i+1}) + \frac{60}{24(2\lambda-1)} h^4 f^{(4)}(x_{i+\lambda})$$

Hence:

$$h^5 \left(S_n^{(5)}(x_{i+}) - f^{(5)}(x_{i+}) \right) =$$
$$-\frac{60\lambda^3}{(\lambda-1)^2(2\lambda-1)} \frac{h^5}{120} f^5(\theta_{4,i}) - \frac{60(2\lambda-3)}{(\lambda-1)^2(2\lambda-1)} \frac{h^5}{120} f^{(5)}(\theta_{2,i})$$
$$+\frac{300}{(\lambda-1)(2\lambda-1)} \frac{h^5}{120} f^{(5)}(\theta_{8,i}) + \frac{300\lambda}{(2\lambda-1)} \frac{h^5}{120} f^{(5)}(\theta_{6,i})$$
$$-h^5 f^{(5)}(\theta_{10,i}) + \frac{60}{\lambda(2\lambda-1)} h\left(S_n^{'}(x_i) - f'(x_i)\right) + \frac{60}{(\lambda-1)(2\lambda-1)}$$
$$h\left(S_n^{'}(x_{i+1}) - f'(x_{i+1})\right) = \frac{5(\lambda^2 - \lambda + 1)}{2(\lambda-1)(2\lambda-1)} \alpha_2 w\left(f^{(5)}; \frac{1}{m}\right) h(S_n^{'}(x_i)$$
$$-f'(x_i)) + \frac{60}{(\lambda-1)(2\lambda-1)} h\left(S_n^{'}(x_{i+1}) - f'(x_{i+1})\right), |\alpha_2| \le 1$$

By using (10), we get (11). The proofs of (12-16) are similar and we only mention that:

$$h^5 S_n^{(5)}(x_{i+\lambda}) = \frac{60(2\lambda+1)}{\lambda^2(2\lambda-1)} f(x_i) - \frac{60}{\lambda^2(\lambda-1)^2(2\lambda-1)} f(x_{i+\lambda}) -$$
$$\frac{60(2\lambda+3)}{(\lambda-1)^2(2\lambda-1)} f(x_{i+1}) + \frac{60}{\lambda(2\lambda-1)} h S_n^{'}(x_i) +$$
$$\frac{60}{(\lambda-1)(2\lambda-1)} h S_n^{'}(x_{i+1}) + \frac{60}{24(2\lambda-1)} h^4 f(x_{i+\lambda})$$

$$h^5 S_n^{(5)}(x_{i-}) = \frac{60(2\lambda+1)}{\lambda^2(2\lambda-1)} f(x_{i-1}) - \frac{60}{\lambda^2(\lambda-1)^2(2\lambda-1)} f(x_{i-1+\lambda}) -$$
$$\frac{60(2\lambda+3)}{(\lambda-1)^2(2\lambda-1)} f(x_i) + \frac{60}{\lambda(2\lambda-1)} h S_n^{'}(x_{i-1}) +$$
$$\frac{60}{(\lambda-1)(2\lambda-1)} h S_n^{'}(x_i) + \frac{60}{24(2\lambda-1)} h^4 f(x_{i-1+\lambda})$$

$$h^3 S_n^{(3)}(x_{i+\lambda}) =$$
$$\frac{3(-12\lambda^3 + 6\lambda^2 + 4\lambda - 3)}{2\lambda^2(\lambda-\frac{1}{2})} f(x_i) + \frac{3(10\lambda^2 + 10\lambda + 3)}{2\lambda^2(\lambda-1)^2(\lambda-\frac{1}{2})} f(x_{i+\lambda}) +$$
$$\frac{3(12\lambda^3 - 30\lambda^2 + 20\lambda - 5)}{2(\lambda-1)^2(\lambda-\frac{1}{2})} f(x_{i+1}) + \frac{3(-6\lambda^2 + 8\lambda - 3)}{2\lambda(\lambda-\frac{1}{2})} h S_n^{'}(x_i) +$$
$$\frac{3(-6\lambda^2 + 4\lambda - 1)}{2(\lambda-1)(\lambda-\frac{1}{2})} h S_n^{'}(x_{i+1}) + \frac{3(6\lambda^2 - 6\lambda + 1)}{48(\lambda-\frac{1}{2})} h^4 f^{(4)}(x_{i+\lambda})$$

$$h^2 S_n^{(2)}(x_{i+\lambda}) =$$

$$\frac{2(\lambda-1)(-4\lambda^2-2\lambda+1)}{\lambda^2} f(x_i) + \frac{2(5\lambda^2-5\lambda+1)}{\lambda^2(\lambda-1)^2} f(x_{i+\lambda}) -$$

$$\frac{2\lambda(-4\lambda^2+10\lambda-5)}{(\lambda-1)^2} f(x_{i+\lambda}) - \frac{2\lambda(-4\lambda^2+10\lambda-5)}{(\lambda-1)^2} f(x_{i+1}) +$$

$$\frac{2(\lambda-1)(-2\lambda+1)}{\lambda} hS_n^{'}(x_i) + \frac{2(-2\lambda+1)}{(\lambda-1)} hS_n^{'}(x_{i+1}) +$$

$$\frac{\lambda(\lambda-1)}{12} h^4 f^{(4)}(x_{i+\lambda})$$

and

$$hS_n^{'}(x_{i+\lambda}) =$$

$$\frac{(\lambda-1)^2(-6\lambda^2-3\lambda+4)}{2\lambda(2\lambda-1)} f(x_i) + \frac{(15\lambda^2-15\lambda+4)}{2\lambda(\lambda-1)(2\lambda-1)} f(x_{i+\lambda}) -$$

$$\frac{\lambda^2(-6\lambda^2+15\lambda-5)}{2(\lambda-1)(2\lambda-1)} f(x_{i+1}) + \frac{(\lambda-1)(-3\lambda+2)}{2(2\lambda-1)} hS_n^{'}(x_i) +$$

$$\frac{\lambda^2(-3\lambda+1)(-2\lambda+1)}{2(2\lambda-1)} hS_n^{'}(x_{i+1}) + \frac{\lambda^2(\lambda+1)}{48(2\lambda-1)} 2 h^4 f^{(4)}(x_{i+\lambda})$$

Theorem 2: Let $f \in C^5[0,1]$ and $S_n(x) \in S_{n,5}^{(2)}$ be a unique spline satisfying the conditions of Theorem 1, then:

$$\left\| S_n^r(x) - f^{(r)}(x) \right\| \le k \, n^{r-5} w\left(f^{(5)}; \frac{1}{n} \right), \quad r = 0,1,2,3,4$$

$$k = \frac{\begin{pmatrix} 57600\lambda^{12} - 228540\lambda^{11} + 345418\lambda^{10} - 213502\lambda^9 + \\ 32668\lambda^8 - 76036\lambda^7 + 244410\lambda^6 - 313247\lambda^5 + \\ 222507\lambda^4 + 86629\lambda^3 + 11505\lambda^2 + 3376\lambda - 1146 \end{pmatrix}}{960\lambda^2(\lambda-1)^2\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2-12\lambda+1)}$$

Proof: For $0 \le y \le 1$, we obtain:

$$A_0(y) + A_1(y) + A_2(y) = 1 \tag{17}$$

Let $x_i \le x \le x_{i+1}$. On using (17) and (7), we obtain:

$$S_n^{(5)}(x) - f^{(5)}(x) = \left(S_n^{(5)}(x_{i+}) - f^{(5)}(x) \right) A_0\left(\frac{x-x_i}{h} \right) +$$

$$\left(S_n^{(5)}(x_{i+\lambda}) - f^{(5)}(x) \right) A_1\left(\frac{x-x_i}{h} \right) + \tag{18}$$

$$\left(S_n^{(5)}(x_{i-}) - f^{(5)}(x) \right) A_2\left(\frac{x-x_i}{h} \right) =$$

$$E_1 + E_2 + E_3$$

From (3) it follows that:

$$|A_0(x)| \le 1, \quad |A_1(x)| \le 1 \text{ and } |A_2(x)| \le 1 \text{ on } 0 \le x \le 1$$

$$f^{(5)}(x) = f^{(5)}(\emptyset_{10,i}) \quad x_i \le \emptyset_{10,i} \le x$$

$$E_1 = \left(S_n^{(5)}(x_{i+}) - f^{(5)}(x) \right) A_0\left(\frac{x-x_i}{h} \right) =$$

$$A_0\left(\frac{x-x_i}{h} \right) (S_n^{(5)}(x_{i+}) - f^{(5)}(\emptyset_{10,i}))$$

On using (2) and $|x - x_i| \le h$ we obtain:

$$|E_1| \le \frac{(120\lambda^6 - 220\lambda^5 + 274\lambda^4 - 166\lambda^3 + 34\lambda^2 + 4\lambda - 6)}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2-12\lambda+1)}$$
$$w\left(f^{(5)}; \frac{1}{n} \right) \tag{19}$$

Similarly:

$$|E_3| \le \frac{\begin{pmatrix} 120\lambda^7 - 400\lambda^6 + 656\lambda^5 - 673\lambda^4 + 372\lambda^3 - \\ 99\lambda^2 + 3\lambda - 3 \end{pmatrix}}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2-12\lambda+1)}$$
$$w\left(f^{(5)}; \frac{1}{m} \right) \tag{20}$$

$$E_2 = \left(S_n^{(5)}(x_{i+\lambda}) - f^{(5)}(x) \right) A_1\left(\frac{x-x_i}{h} \right) =$$
$$A_1\left(\frac{x-x_i}{h} \right) (S_n^{(5)}(x_{i,\lambda}) - f^{(5)}(\emptyset_{10,i})) \tag{21}$$

$$|E_2| \le \frac{(120\lambda^6 - 220\lambda^5 + 274\lambda^4 - 166\lambda^3 + 34\lambda^2 + 4\lambda - 6)}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)(12\lambda^2-12\lambda+1)}$$
$$w\left(f^{(5)}; \frac{1}{n} \right) \tag{22}$$

Putting (19-22) in (18) we obtain:

$$|S_n^{(5)}(x) - f^{(5)}(x)| \le$$
$$\frac{\begin{pmatrix} 360\lambda^7 - 960\lambda^6 + 1424\lambda^5 - 1279\lambda^4 + \\ 606\lambda^3 - 114\lambda^2 - 19\lambda + 9 \end{pmatrix}}{8\lambda^2(\lambda-1)\left(\lambda-\frac{1}{2}\right)^2(12\lambda^2-12\lambda+1)} w(f^{(5)}; \frac{1}{n}) \tag{23}$$

This proves Theorem 2 for r = 5. To prove Theorem 2 for r = 4, since $S_n^{(4)}(x_{i+\lambda}) - f^{(4)}(x_{i+\lambda}) = 0$:

$$S_n^{(4)}(x) - f^{(4)}(x) = \int_{x_{i+\lambda}}^{x} (S_n^{(5)}(t) - f^{(5)}(t))dt$$

On using (23) we obtain:

$$\left|S_n^{(4)}(x) - f^{(4)}(x)\right| \le$$

$$\frac{\left(\begin{array}{l}360\lambda^7 - 960\lambda^6 + 1424\lambda^5 - 1279\lambda^4 + \\ 606\lambda^3 - 114\lambda^2 - 19\lambda + 9\end{array}\right)}{8\lambda^2(\lambda-1)\left(\lambda-\dfrac{1}{2}\right)(12\lambda^2 - 12\lambda + 1)} hw\left(f^{(5)}; \frac{1}{n}\right) \quad (24)$$

This proves Theorem 2 for r = 4. To prove Theorem 2 for r = 3, since:

$$S_n^{(3)}(x) - f^{(3)}(x) = \int_{x_{i+\lambda}}^{x} (S_n^{(4)}(t) - f^{(4)}(t))dt + S_n^{''}(x_{i+\lambda}) - f'''(x_{i+\lambda})$$

On using (14) and (24) we obtain:

$$\left|S_n^{(3)}(x) - f^{(3)}(x)\right| \le$$

$$\frac{\left(\begin{array}{l}960\lambda^9 - 3000\lambda^8 + 10436\lambda^7 - 20704\lambda^6 + 28238\lambda^5 - \\ 25156\lambda^4 + 11994\lambda^3 - 2322\lambda^2 - 376\lambda + 189\end{array}\right)}{\lambda^2(\lambda-1)\left(\lambda-\dfrac{1}{2}\right)^2(12\lambda^2 - 12\lambda + 1)} \quad (25)$$

$$\frac{h^2}{160} w\left(f^{(5)}; \frac{1}{n}\right)$$

Which proves Theorem 2 for r = 3. To prove Theorem 2 for r = 2, since:

$$S_n^{''}(x) - f''(x) = \int_{x_{i+\lambda}}^{x} (S_n^{(3)}(t) - f^{(3)}(t))dt + S_n^{''}(x_{i+\lambda}) - f''(x_{i+\lambda})$$

On using (15) and (25) we obtain:

$$\left|S_n^{''}(x) - f''(x)\right| \le$$

$$\frac{\left(\begin{array}{l}480\lambda^{11} - 560\lambda^{10} - 2988^9 + 18810\lambda^8 - \\ 48056\lambda^7 + 75451\lambda^6 - 81111\lambda^5 + 55783\lambda^4 - \\ 21639\lambda^3 + 2867\lambda^2 - 942.5\lambda - 286.5\end{array}\right)}{\lambda^2(\lambda-1)^2\left(\lambda-\dfrac{1}{2}\right)^2(12\lambda^2 - 12\lambda + 1)} \quad (26)$$

$$\frac{h^3}{240} w\left(f^{(5)}; \frac{1}{n}\right)$$

Which proves Theorem 2 for r = 2. To prove Theorem 2 for r = 1, since:

$$S_n^{'}(x) - f'(x) = \int_{x_{i+\lambda}}^{x} (S_n^{(2)}(t) - f^{(2)}(t))dt + S_n^{'}(x_{i+\lambda}) - f'(x_{i+\lambda})$$

On using (16) and (26) we obtain:

$$\left|S_n^{'}(x) - f'(x)\right| \le$$

$$\frac{\left(\begin{array}{l}57600\lambda^{12} - 228540\lambda^{11} + 345418\lambda^{10} - 213502^9 + \\ 32668\lambda^8 - 76036\lambda^7 + 244410\lambda^6 - 313247\lambda^5 + \\ 222507\lambda^4 - 86629\lambda^3 + 11505\lambda^2 - 3376\lambda - 1146\end{array}\right)}{\lambda^2(\lambda-1)^2\left(\lambda-\dfrac{1}{2}\right)^2(12\lambda^2 - 12\lambda + 1)} \quad (27)$$

$$\frac{h^3}{960} w\left(f^{(5)}; \frac{1}{n}\right)$$

This proves Theorem 2 for r = 1. To prove Theorem 2 for r = 0, since $S_n^{'}(x_{i+\lambda}) - f'(x_{i+\lambda}) = 0$:

$$S_n(x) - f(x) = \int_{x_{i+\lambda}}^{x} (S_n^{'}(t) - f'(t))dt$$

$$\left|S_n(x) - f(x)\right| \le k \ h^5 w\left(f^{(5)}; \frac{1}{n}\right)$$

$$\left|S_n(x) - f(x)\right| \le k \ m^{-5} w\left(f^{(5)}; \frac{1}{n}\right)$$

This completes the proof of Theorem 2.

RESULTS AND DISCUSSION

Numerical results: The results obtained, the existence uniqueness and error bounds for generalize (0,4) lacunary interpolation by qunatic spline.

CONCLUSION

These generalize are prefer to interpolation by quantic spline to the use (0,4).We also can use this idea to generalize for different lacunay type for example (0,2),(0,3),…etc.

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