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Reliability of Simple 3-Dimnesional Consecutive k-Out-of-n: F Systems

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Abstract: Problem statement: In this study, the reliability of some special cases of a 3-dimnesional consecutive k-out-of-n: F system and k-within consecutive (2,2,2) -out-of- (m,2,2): F system were discussed. **Approach:** The 3 dimensional systems is a generalization of 1 and 2 dimensional, systems. Projections from 3 dimensional systems to 2 or 1 dimension as special cases is very helpful to find the reliability of these 3 dimensional systems. **Results:** Many reliability expressions for the 3-dimensional consecutive (k_1,k_2,k_3) -out-of- (m,n,ℓ): F system for some special values of k_1, k_2, k_3, m, n, ℓ are presented. Further, the reliability of k-within (2, 2, 2)-out-of-(m,2,2): F system using Markov chains is considered. **Conclusion/Recommendations:** In general, it is difficult to find the reliability of (k_1,k_2,k_3) -out-of- (m,n,ℓ): F system, we studied special cases of this system and recommend generalizing the result for any value of k_1, k_2, k_3, m, n, ℓ. This study presented the reliability formulas of simple 3D systems using results of consecutive k-out-of-n: F systems and 2D consecutive k-out-of-n: F systems and Markov chains.

Key words: (1, 2 and 3) D-consecutive k-out-of-n: F system, k-within (k_1,k_2,k_3) -out-of- (m,n,ℓ): F system, Markov chains

INTRODUCTION

The consecutive k-out-of-n: F system has been extensively studied in recent years (Bollinger and Salvia, 1982; Chen and Hwang, 1985; Derman *et al*., 1982; Shantikumar, 1982; Cluzeau and Keller, 2008; Lambiris and Papastavridis, 1985). The system is specified by n, the number of components, where each component either functions or fails. The system fails if at least k consecutive components fail. The 2D-consecutive k-out-of-n: F system was introduced by Salvia and Lasher (1992) by generalizing the notion of the consecutive k-out-of-n: F system. A 2D-consecutive k-out-of-n: F system (denoted by $k^2|n^2$: F) was defined as a square grid of side n (containing n^2components). The system fails if there is at least one square of side k ($1 \le k \le n$) that contains all failed components. Zuo (1993) proposed a more general model of 2D- consecutive k-out-of-n: F system (denoted by $k_1k_2|$ mn: F). It is a rectangular grid of dimension $m \times n$ (containing m.n components). The system fails if there is at least one rectangle grid of dimension $k_1 \times k_2$($1 \le k_1,k_2 \le m,n$) that contains all failed components (Yamamoto and Akiba, 2005). Boehme *et al*. (1992) and Yamamoto *et al*. (2008) defined a consecutive (k_1,k_2) or (k_1,k_2) -out-of- (m,n): F system. In this case the system fails if at least one rectangle of dimension $k_1 \times k_2$ or $k_2 \times k_1$that contains all failed components occurs. Similarly the 3D- consecutive k-out-of-n: F system (denoted by $k^3|n^3$: F) is defined as a cube of side n containing n^3components. The system fails if there is at least one cube of side k ($1 \le k \le n$) that contains all failed components (Akiba and Yamamoto, 2002; Akiba *et al*., 2004). Also more general model of 3D- consecutive k-out-of-n: F systems (denoted by $k_1k_2k_3|$mnl: F). It is a cuboid of dimension $m \times n \times \ell$ (containing m.n. ℓ components). The system fails if there is at least one cuboid of dimension $k_1 \times k_2 \times k_3$($1 \le k_1,k_2,k_3, \le m,n,l$) that contains all failed components. This study gives the reliability of simple 3D systems using results of consecutive 1 and 2 D- consecutive k-out-n: F system in addition to a special case of 3D- dimension system which is k-within (2,2,2)-out-of-(m,2,2): F system, this system consists of (m,2,2) cuboid, the system fails if there is at least k failed components in any (2,2,2) cuboid.

MATRIALS AND METHODS

The 3-dimensional systems is a generalization of 1 and 2-dimensional, system, projections from 3-dimensional systems to 2 or 1 dimension as special cases is very helpful to find the reliability of special cases of the 3 dimensional systems. This study gives the reliability of some special cases of the 3-dimensional systems. To describe the problem, we introduce the following notations and assumptions.

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Notations:

- p,q: Components reliability (unreliability) of component where p+q = 1
- $g_k(n,j)$: Configuration of no consecutive k failed components from j through the all n components which equal $\sum_{i=0}^{gilb[j|k]}(-1)^i\binom{n-ik}{j-ik}\binom{n-j+1}{i}$
- α(m,j): Configuration of no connected 2 components in (m,2) system.
- R(k;n;p): The reliability of a consecutive k-out-of-n: F system
- $R((k_1,k_2);(m,n);p)$: The reliability of a consecutive (k1,k2)-out-of-(m,n): F system
- $R((k_1,k_2,k_3);(m,n,l);p)$: The reliability of a consecutive (k1,k2,k3)-out-of-(m,n,l): F system
- $R(k-(2,2,2);(m,2,2);p)$: The reliability of a k-within (2,2,2)-out-of- (m,2,2): F system
- S_r: Random variable count the failed component of cuboid $(2,2,2)$ in the r^{th} layer
- P_k: Markov transition probability matrix
- $P_k(ij)$: Element of transition probability matrix with status i and j

Assumption:

- Each component and the whole system can only be either functioning or failed
- All components are mutually s-independent

Lemma: For the k-within (2,2,2)-out-of-(m,2,2): F system, if i,j the statuses of Markov chains, the transition probability matrix denoted P_k of the system will be:

$$P_k(ij)=\begin{cases}\binom{4}{j}p^{4-j}q^j & i+j<k\\ 0 & i+j\geq k\end{cases}$$

and:

$$R(k-(2,2,2);(m,2,2);p)=\sum_{j=0}^{4}P_k^m(0j)$$

Proof: If i+j<k then the number of failed components in the layer r+l does not depend on the number of failed components in the layer r, so:

$$P_k(ij)=P(S_{r+1}=jS_r=i)=\frac{P(S_{r+1}=j,S_r=i)}{P(S_r=i)}$$
$$=\frac{\binom{4}{j}\binom{4}{i}p^{8-(i+j)}q^{(i+j)}}{\binom{4}{i}p^{4-i}q^i}$$
$$=\binom{4}{j}p^{4-j}q^j$$

If i+j≥k the system will fail, so:

$$P(S_r=i,S_{r+1}=j)=0\Rightarrow P_k(ij)=0$$

Hence; the reliability of the system will be of the form:

$$R(k\text{-}(2,2,2);\ (m,2,2);p)=\sum_{j=0}^{4}P_k^m(0j)$$

RESULT

The reliability formulas of simple 3D systems: The reliability of a consecutive (k,1,1)-out-of-(m,l,l): F system is the same as the reliability of consecutive k-out-of-n: F system. Therefore from (Zuo, 1993) if k≤m,n, ℓ :

$$R((k,1,1,);(m,1,1,);p)=R(k;m;p)$$
$$=\sum_{j=0}^{m-gilb[m|k]}g_k(m,j)p^{m-j}q^j \quad (1)$$

Also:

$$R((1,k,1);(1,n,1);p)=R(k;n;p)$$
$$=\sum_{j=0}^{n-gilb[n|k]}g_k(n,j)p^{n-j}q^j \quad (2)$$

and:

$$R((1,1,k);(1,1,l);p)=R(k;l;p)$$
$$=\sum_{j=0}^{l-gilb[l|k]}g_k(l,j)p^{l-j}q^j \quad (3)$$

A consecutive (l,l,l)-out-of-(m,n, ℓ): F system reliability is the same as the reliability of a series system consisting of m.n.l components. Therefore:

$$R((1,1,1);(m,n,l);p)=R(1;mn\ell;p)=p^{m.n.\ell} \quad (4)$$

The reliability of a consecutive (m,n, ℓ)-out-of-(m,n, ℓ): F system is the same as the reliability of a parallel system consisting of m.n. ℓ components. Therefore:

$$R((1,1,1);(m,n,\ell);p) = R(mn\ell;mn\ell;p) = 1-q^{m.n.\ell} \quad (5)$$

The reliability of a consecutive (k,l,l)-out-of-(m,n, ℓ): F system. In this case the system consists of *l* parallel (m,n)-matrix, any (m,n)-matrix is considered as a consecutive (k,l)-out-of-(m,n): F system. In accordance with our definition, the system will fail if at least one row of any (m,n)-matrix includes k-consecutive failed elements occurs. Then the system operating if all of the rows (consider as a consecutive k-out-of-m: F system) are operating. Since the elements fail independently, the rows and the matrices fail independently.

The probability that the (m,n) matrix does not fail is equal to the product of the probabilities that the rows of this matrix do not fail, therefore:

$$R((k,1);(m,n);p) = [R(k;m;p)]^{n} \quad (6)$$

and:

$$R((k,1,1);(m,n,\ell);p) = [R(k;m;p)]^{n.\ell} \quad (7)$$

Analogously for the consecutive (l,k,l)-out-of-(m,n, ℓ): F system, the reliability function:

$$R((1,k,1);(m,n,\ell);p) = [R(k;n;p)]^{m.\ell} \quad (8)$$

And for the consecutive (l,l,k)-out-of-(m,n, ℓ): F system, the reliability function:

$$R((1,1,k);(m,n,\ell);p) = [R(k;\ell;p)]^{m.n} \quad (9)$$

A consecutive (m,k,l)-out-of-(m,n, ℓ): F system, in this case, the system consists of ℓ parallel (m,n,)-matrix. Any matrix is considered as a consecutive (m,k)-out-of-(m,n): F system. In accordance with our definition, the system will fail if k-consecutive columns each including m failed elements occur of any (m,n,)-matrix. Then we consider the column as new "element" with failure probability q^m and reliability $1-q^m$ and any (m,n) matrix as a consecutive k-out-of-m: F system, having the reliability $R(k;l;1-q^m)$, therefore:

$$R((m,k,1);(m,n,\ell);p) = [R(n,k;1-q^{m})]^{\ell} \quad (10)$$

Analogously, for consecutive (k,n,l)-out-of-(m,n, ℓ): F system, the reliability will be:

$$R((k,n,1);(m,n,\ell);p) = [R(k,m;1-q^{n})]^{\ell} \quad (11)$$

and, for consecutive (m,l,k)-out-of-(m,n, ℓ): F system, the reliability will be:

$$R((m,1,k);(m,n,\ell);p) = [R(k,n;1-q^{m})]^{n} \quad (12)$$

and, for consecutive (k,l,l)-out-of-(m,n, ℓ): F system, the reliability will be:

$$R((k,1,\ell);(m,n,\ell);p) = [R(n,k;1-q^{\ell})]^{n} \quad (13)$$

and, for consecutive (l,k, ℓ)-out-of-(m,n, ℓ): F system, the reliability will be:

$$R((1,k,\ell);(m,n,\ell);p) = [R(m,k;1-q^{\ell})]^{m} \quad (14)$$

and, for consecutive (l,n,k)-out-of-(m,n, ℓ): F system, the reliability will be:

$$R((1,n,k);(m,n,\ell);p) = [R(m,k;1-q^{n})]^{m} \quad (15)$$

A consecutive (m,n,k)-out-of-(m,n, ℓ): F system ($k \le \ell$). The system fails if k consecutive (m,n)-matrices each including m.n failed elements. Then we can consider-matrices as new "element" with failure probability $q^{m.n}$ and reliability $1-q^{m.n}$, the system equivalents to consecutive k-out-of- ℓ : F system having the reliability R (ℓ ,k;$1-q^{m.n}$).

Analogously, for consecutive (m,k, ℓ)-out-of-(m,n, ℓ): F system ($k \le n$), the system equivalents to consecutive k-out-of-n: F system having the reliability R(n, k;, $1-q^{m.\ell}$) and for consecutive (k,n, ℓ)-out-of-(m,n, ℓ)-: F system ($k \le m$), the system equivalents to consecutive k-out-of-m: F system having the reliability R(m, k; $1-q^{n.\ell}$).

A consecutive (1,2,1)-or-(2,1,1)-out-of-(m,2,1): F system. The system consists of (m,2)-matrix i.e., the reliability of the system is the same as (1,2)-or-(2,1)-out-of-(m,2): F system, because it fails if 2 connected components fail, therefore from (El-Sayed, 1998):

$$\begin{aligned} & R((1,2,1)-or-(2,1,1);(m,2,1);p) \\ & = R((1,2)-or-(2,1);(m,2);p) \\ & = \sum_{j=0}^{m} \alpha(m,j) p^{2m-j} q^{j} \end{aligned} \quad (16)$$

$$\alpha(m,j) = \begin{cases} 1 & j=0 \\ 2m & j=1 \\ 2 & j=m \\ \alpha(m-1,j) + 2\sum_{i=1}^{j} \alpha(m-i-1,j-i) & 1<j<m \\ 0 & \text{otherwise} \end{cases} \quad (17)$$

Also:

$$R\begin{pmatrix}(1,2,1)-\text{or}-(2,1,1);\\(2,n,p);p\end{pmatrix}=R\begin{pmatrix}(1,2)-\text{or}-\\(2,1);(n,2);p\end{pmatrix}=\sum_{j=0}^{n}\alpha(n,j)p^{2n-j}q^{j} \quad (18)$$

and:

$$R\begin{pmatrix}(1,2,1)-\text{or}\\-(2,1,1);(1,2,l);p\end{pmatrix}=R\begin{pmatrix}(1,2)-\text{or}\\-(2,1);(l,2);p\end{pmatrix}=\sum_{j=0}^{l}\alpha(l,j)p^{2l-j}q^{j} \quad (19)$$

Reliability of k-within (2,2,2)-out-of-(m,2,2): F system.

According the lemma 1 we can directly find the transitive probability matrix and compute the reliability of the system.

Example: Compute $R(4-(2,2,2);(3,2,2);p)$:

$$P_4=\begin{pmatrix}p^4 & 4p^3q & 6p^2q^2 & 4pq^3\\p^4 & 4p^3q & 6p^2q^2 & 0\\p^4 & 4p^3q & 0 & 0\\p^4 & 0 & 0 & 0\end{pmatrix}$$

$$P_4^3(00)=p^{12}+8p^{11}q+28p^{10}q^2+56p^9q^3$$

$$P_4^3(01)=4p^{11}q+32p^{10}q^2+114p^9q^3+208p^8q^4$$

$$P_4^3(02)=6p^{10}q^2+48p^9q^3+132p^8q^4+168p^7q^5$$

$$P_4^3(03)=4p^9q^3+16p^8q^4+24p^7q^5+16p^6q^2$$

$$P_4^3(03)=4p^9q^3+16p^8q^4+24p^7q^5+16p^6q^2$$

$$R(4-(2,2,2);(3,2,2);p)=\sum_{j=0}^{4}P_4^3(0j)$$

$$R(4-(2,2,2);(3,2,2);p)=p^{12}+12p^{11}q+66p^{10}q^2+222p^9q^3+356p^8q^4+192p^7q^5+16p^6q^2$$

DISCUSSION

In general, it is difficult to find the reliability of (k_1,k_2,k_3) -out-of- (m,n,ℓ): F system. Thus, in this article, we found the reliability of some special cases in an explicit form. Projections from 3 dimensional systems to 2 or 1 dimension as special cases is very helpful to find the reliability of these 3 dimensional systems. Also we found the reliability of k-within (2,2,2)-out-of-(m,2,2): F system using Markov Chains.

CONCLUSION

In this study, we concluded the reliability of some special cases of (k_1,k_2,k_3)-out-of-(m,n,ℓ): F system. Also, the reliability of k-within (2,2,2)-out-of-(m,2,2): F system is derived easily using Markov Chains.

REFERENCES

Akiba, T. and H. Yamamoto, 2002. Upper and lower bounds for 3-dimensional k-within consecutive-(r_1, r_2, r_3)-out-of-(n_1,n_2,n_3): F system. IEIC Tech. Rep., 102: 1-6. DOI: 10.1142/9789812702685_0002

Akiba, T., H. Yamamoto and Y. Kainuma, 2004. Reliability of a 3-dimensional adjacent triangle: F triangular lattice system. Proceedings of the 5th Asia Pacific Industrial Engineering and Management Systems Conference, Aug. 27-27, APIEMS, Gold Coast, Australia, pp: 1-10.

Boehme, T.K., A. Kossow and W. Press, 1992. A generalization of consecutive k-out-of-n: F system. IEEE Trans. Reliab., 41: 451-457.

Bollinger, R.C. and A.A. Salvia, 1982. Consecutive k-out-of-n: F system networks. IEEE Trans. Reliab., 31: 53-56. DOI: 10.1109/TR.1982.5221227

Chen, R.W. and F.K. Hwang, 1985. Failure distributions of consecutive k-out-of-n: F systems. IEEE Trans. Reliab., 34: 338-341.

Cluzeau, T. and J. Keller, 2008. An efficient algorithm for computing the reliability of consecutive-k-out-of-n: F systems. IEEE Trans. Reliab., 57: 53-56.

Derman, C., G.J. Lieberman and M. Ross, 1982. On the Consecutive k-out-of-n: F system. IEEE Trans. Reliab., 31: 57-63.

El-Sayed, E.M., 1998. Algorithm for the reliability of (1,2) or (2,1)-out-of- (n,2): F systems. J. Egypt. Math. Soc., 2: 169-173.

Lambiris, M. and S. Papastavridis, 1985. Exact reliability formula for linear and circular consecutive k-out-of-n: F system. IEEE Trans. Reliab., 34: 124-126.

Salvia, A.A. and W.C. Lasher, 1992. 2-dimensional consecutive k-out-of-n: F system models. IEEE Trans. Reliab., 41: 575-578.

Shantikumar, J.G., 1982. Recursive algorithm to evaluate the reliability of consecutive k-out-of-n: F system networks. IEEE Trans. Reliab., 31: 84-87.

Yamamoto, H. and T. Akiba, 2005. A recursive algorithm for the reliability of a circular connected-(r,s)-out-of-(m,n): F lattice system. Comput. Ind. Eng., 49: 21-34. DOI: 10.1016/j.cie.2005.01.015

Yamamoto, H., T. Akiba, H. Nagatsuka and Y. Moriyama, 2008. Recursive algorithm for the reliability of a connected-(1,2)-or-(2,1)-out-of-(m,n): F lattice system. Eur. J. Oper. Res., 188: 854-864. DOI: 10.1016/j.cie.2005.01.015

Zuo, M.J., 1993. Reliability and design of 2-imensional consecutive k-out-of-n: F system. IEEE Trans. Reliab., 39: 488-490. DOI: 10.1109/24.257836