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Computing the Moments of Order Statistics from Independent Non – Identically Distributed Burr Type XII Random Variables

A.A. Jamjoom
Department of Mathematics, Girls College of Education, Jeddah, Saudi Arabia

Abstract. In this paper, we derive a recurrence relation for computing all single moments of all order statistics arising from independent but not identically distributed Burr type XII random variables.

Keywords: Independent non – identical Variates; recurrence relations; order statistics, Moments, Burr distribution

INTRODUCTION

Let $X_1, X_2, ..., X_n$ be independent random Variables having cumulative distribution functions i.e. $F_1(x), F_2(x), ..., F_n(x)$ and probability density function , respectively.

Let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the order Statistics obtained by arranging the n $X_i's$ in increasing order of magnitude . Bapat and Beg [1] have shown that the CDF of the rth order Statistics $X_{r:n} (l \le r \le n)$ is conveniently expressed in terms of permanents as follows

$$F_{r:n}(x) = \sum_{i=r}^{n} \frac{1}{i!(n-r)!} Per[\underset{i}{F(x)} \quad \underset{n-i}{1-F(x)}], \ -\infty \langle \ x \ \langle \ \infty \quad (1)$$

where $F(x)$ and $1-F(x)$ denote the column vectors $\left(F_1(x), F_2(x), ..., F_n(x)\right)'$ and $\left(1-F_1(x), 1-F_2(x), ..., 1-F_n(x)\right)'$ respectively.

Moreover if $a_1, a_2, ...$ are column vectors, then

$$[\underset{i_1}{a_1} \quad \underset{i_2}{a_2}],$$

will denote the matrix obtained by taking i_1 copies of a_1, i_2 copies of a_2 and so on. Also, in (1) , Per (A) denotes the permanent of a square matrix A ; which is defined similarly as the determinant of A except that all terms in the expansion have a positive sign, i.e.

$$Per\,A = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$$

where S_n is the set of permutations of 1,2,...,n[2,3,4] .

In the last few years much attention had been paid to order statistics from independent nonidentically distributed variables i.ni.d[5,6,7,8] .

Derivation of recurrence relations for single moments of order statistics from i.ni.d available samples found in the literature have taken two directions, the work initiated by Balakrishnan [5,6] and that of Barakat and Abdelkader [9] In Balakrishnan 's work[5,6] , a linear relation between the PDF and CDF of the distribution , if exists, is exploited and then one has to go through messy calculations using integration by parts to get the result. Application of this method was done on many distributions such as: exponential[5], right-truncated exponential[6] doubly-truncated exponential and logistic distribution[10,11] , power function distribution[7] , Pareto and doubly-truncated Pareto distributions [12]. All of these results were obtained by exploiting a basic differential equation satisfied by the distribution under consideration. For example: the differential equation satisfied by the PDF and CDF of exponential distribution is

$$f_i(x) = \frac{1}{\theta_i}\left\{1-F_i(x)\right\}, \ x \ge 0 \ , \ \theta_i \ \rangle \ 0 \quad i=1,2,...,n,$$

for Pareto distributions it is

$$F_i(x) = 1 - \frac{x}{\upsilon_i} f_i(x) \quad i=1,2, ..., n$$

and for power function distributions it is

$$x f_i(x) = \upsilon_i \ F_i(x) \ , \ 0 \le x \le 1$$
$$\upsilon_i \ \rangle \ 0 \ ,$$
$$i = 1,2,...,n$$

However, most of these recurrence relations show that it is enough to evaluate the kth moment of a single order statistics in a sample of size n, if these moments in samples of size less than n are already available. The kth moments of the remaining n – 1 order statistics can then be determined by repeated use of these recurrence relations.

Barakat and Abdelkader[9] generalized their procedure initiated in (2000)[13] to any d.f. and expressed the kth moment of the rth order statistics $\mu_{r:n}^{(k)}\,(k=1,2,\ldots),\left(1\le r\le n\right)$ of a sample of size n purely in terms of the kth moments of the maximum order statistics or of the minimum order statistics from samples of size up to n of all possible subsamples of the given samples. This in fact simplifies the recursive computation of the single moments of (i.nid) order statistics.

Application of Barakat and Abdelkader 's method[9] started in fact in (2000)[13] when they first applied it to calculate single moments of non-identically distributed Weibull random and in the year (2004) to Erlang distribution by[14].

The advantages of their procedure can be simply described as follows: first there is no conditions imposed on the CDF. and PDF. of the underlying distribution, i.e. whether they are related on not ; secondly $\mu_{r:n}^{(k)}\left(1\le r\le n\right)$ obtained by their method is purely expressed in terms of the kth moments of maximums and the minimums of all possible subsamples of the given sample.

In this paper we consider the case where the r.v.s $X_i,\ i=1,2,\ldots,n$ are independent and non identical having Burr type XII distribution with CDF.

$$F_i(x)=1-\left(1+x^2\right)^{-m_i},\ x\ge 0,\qquad c,m\ge 1 \tag{2}$$

For $i=1,2,\ldots,n$, where c and m_i are shape parameters[15,16,17,18,19] . We consider Burr type XII distribution since it is widely used in approximation, and as failure rate model[20] and also in predication[21,22], and in many other fields [23,24,25,26] . It has the advantage of being used in approximating distributions of rather complicated PDF's (i.e. intractable distributions)[27,28,29,30,31]. Burr distribution, also known as Lomax at c = 1 or compound Weibull or Weibull Gamma distribution[32]. At m = 1, the Burr distribution reduces to loglogistic or Weibull- Exponential distribution Al–shboul and Khan[33,34,35].

In the next section we derive the kth moment of the largest order statistics $\mu_{n:n}^{(k)}=E\left(X_{n:n}^{k}\right)$ and smallest order statistics $\mu_{1:n}^{(k)}=E\left(X_{1:n}^{k}\right)$. Moreover, a recurrence relation is introduced which will enable one to compute the kth moments of all order statistics $\left(\mu_{r:n}^{(k)},\ for\ all\ r\le n\right)$ in a simple manner by using only the kth moments of the maximum.

Relations for single moments: We shall present some recurrence relations for the single moments of order statistics obtained from Burr type XII distributions.

Relation 2.1: For $n=1,2,\ldots\ and\ \ k=1.2,\ \ldots..$

$$\mu_{1:n}^{(k)}=\frac{k}{c}I_n \tag{3}$$

$$\mu_{n:n}^{(k)}=\frac{k}{c}\sum_{j=1}^{n}(-1)^{j+1}I_j$$

where

$$I_i=\sum_{1\le i\langle 1\langle i_2\langle\ldots\ \langle i\ \le n}\ \sum\ \sum\ \ldots\ \sum B\left(\sum_{i=1}^{j}m_{i_j}-\frac{k}{c},\frac{k}{c}\right) \tag{5}$$

and B (c,k) is the regular beta function define by

$$B\left(c,k\right)=\frac{\Gamma(c)\,\Gamma(k)}{\Gamma\left(c+k\right)}.$$

Proof

Since $x_{i0}=\inf\{x:F_i(x)\succ 0\}\ge 0, i=1,2,\ldots,n$, then by definition of moments we have:

$$\begin{aligned}\mu^{(k)}&=E\left(X^k\right)\\&=\int_0^\infty x^k f(x)\,dx\\&=\int_0^\infty x^k\,dF(x)\\&=-\int_0^\infty x^k\,d\,(1-F(x)),\end{aligned} \tag{6}$$

Integrating by part gives

$$\mu^{(k)}=k\int_0^\infty x^{k-1}\left[1-F(x)\right]dx \tag{7}$$

This equation was obtained by Galambos[36]. Then the kth moment of the smallest is

$$\therefore \mu_{1:n}^{(k)} = k \int_0^\infty x^{k-1} \left(1 - F_{1:n}(x)\right) dx \tag{8}$$

where $F_{1:n}(x)$ is the CDF of the smallest order statistics from independent not identically distributed random variables defined by

$$F_{1:n}(x) = 1 - \prod_{i=1}^{n} \left(1 - F_i(x)\right) \tag{9}$$

$$\left(1 - F_{1:n}(x)\right) = \prod_{i=1}^{n} \left(1 - F_i(x)\right) \tag{10}$$

substituting (10) in (8) we get

$$\mu_{1:n}^{(k)} = k \int_0^\infty x^{k-1} \prod_{i=1}^{n} \left(1 - F_i(x)\right) dx \tag{11}$$

Now substituting (2) in (11) we get

$$\begin{aligned} \mu_{1:n}^{(k)} &= k \int_0^\infty x^{k-1} \prod_{i=1}^{n} \left(1 + x^c\right)^{-m_i} dx \\ &= k \int_0^\infty x^{k-1} \left(1 + x^c\right)^{-\sum_{i=1}^{n} m_i} dx \end{aligned} \tag{12}$$

upon using

$$\int_0^\infty x^{k-1} \left(1 + x^c\right)^{-\alpha} dx = \frac{1}{c} B\left(\alpha - \frac{k}{c}, \frac{k}{c}\right) \tag{13}$$

where $B(c, k)$ is the regular beta function

$$\therefore \mu_{1:n}^{(k)} = \frac{k}{c} B\left(\sum_{i=1}^{n} m_i - \frac{k}{c}, \frac{k}{c}\right)$$

which can be written as

$$\mu_{1:n}^{(k)} = \frac{k}{c} I_n$$

where

$$I_n = B\left(\sum_{i=1}^{n} m_i - \frac{k}{c}, \frac{k}{c}\right)$$

which can also be written as

$$I_n = \sum_{1 \le i_1 \langle i_2 \langle \ldots \langle i_n \le n} \sum \ldots \sum B\left(\sum_{i=1}^{n} m_{i_n} - \frac{k}{c}, \frac{k}{c}\right)$$

where the symbol $\sum_{1 \le i_1 \prec i_2 \prec} \sum_{\ldots \prec i_n} \ldots \sum_{\le n} l_{i_n}$ denote to the sum of the l_n from all possible subsamples of size n (which is one sample in this case) of the given sample . The proof of (4) follows:

$$\mu_{n:n}^{(k)} = k \int_0^\infty x^{k-1} \left[1 - F_{n:n}(x)\right] dx$$

where $F_{n:n}(x)$ the CDF of the largest order statistics from independent not identically distributed random variable defined by

$$F_{n:n}(x) = \prod_{i=1}^{n} F_i(x)$$

and for Burr Type XII it is

$$F_{n:n}(x) = \prod_{i=1}^{n} \left[1 - \left(1 + x^c\right)^{-m_i}\right] \tag{14}$$

$$\therefore \mu_{n:n}^{(k)} = k \int_0^\infty x^{k-1} \left[1 - \prod_{i=1}^{n} \left[1 - \left(1 + x^c\right)^{-m_i}\right]\right] dx$$

$$= k \int_0^\infty x^{k-1} \left[\begin{array}{l} \sum_{i=1}^{n} \left(1 + x^c\right)^{-m_i} - \sum_{1 \le i_1 \langle i_2 \le n} \sum \left(1 + x^c\right)^{-\left(m_{i_1} + m_{i_2}\right)} \\ + \sum_{1 \le i_1 \langle i_2 \langle i_3 \le n} \sum \sum \left(1 + x^c\right)^{-\left(m_{i_1} + m_{i} + m_{i_3}\right)} \\ + \ldots\ldots\ldots\ldots \\ + (-1)^{n+1} \left(1 + x^c\right)^{-\sum_{i=1}^{n} m_i} \end{array}\right] dx$$

Using (13) we get

$$\mu_{n:n}^{(k)} = \frac{k}{c}\left[\begin{array}{l} \sum_{i=1}^{n} B\left(\sum_{i=1}^{n} m_i - \frac{k}{c}, \frac{k}{c}\right) \\ - \sum_{1 \le i_1 \langle i_2 \le n} \sum B\left(m_{i_1} + m_{i_2} - \frac{k}{c}, \frac{k}{c}\right) \\ + \sum_{1 \le i_1 \langle i_2 \langle i_3 \le n} \sum \sum B\left(m_{i_1} + m_{i_2} + m_{i_3} - \frac{k}{c}, \frac{k}{c}\right) \\ + \ldots\ldots. \\ + (-1)^{n+1} B\left(\sum_{i=1}^{n} m_i - \frac{k}{c}, \frac{k}{c}\right) \end{array}\right]$$

$$\mu_{n:n}^{(k)} = \frac{k}{c} \sum_{j=1}^{n} (-1)^{j+1} I_j$$

where

$$I_j = \sum_{1 \le i_1 \langle i_2 \langle \ldots \langle i_j \le n} \sum \ldots \sum \beta\left(\sum_{i=1}^{j} m_{i_j} - \frac{k}{c}, \frac{k}{c}\right)$$

where the symbol $\sum_{1 \le i_1 \langle i_2 \langle} \sum_{\ldots \langle i_j} \ldots \sum_{\le n} l_{i_j}$ denote to the sum of the l_j from all possible subsamples of size j of the given sample (which are $\binom{n}{j}$ samples in this case) .

Relation 2.2. For $r = 1, 2, \ldots., n$ and $k = 1, 2, \ldots.$

$$\mu_{r:n}^{(k)} = \mu_{r-1:n}^{(k)} + J_{r:n}^{(k)} \quad (15)$$

where

$$J_{r:n}^{(k)} = \sum_{j=1}^{r} (-1)^{j-1} \frac{k}{c} a_j I_{n-r+j} \quad (16)$$

where

$$a_j = \frac{(n-r+j)!}{(n-r+1)!(j-1)!}$$

and the sequence $\{I_j\}_{j=1}^{j=r}$ is defined in relation (3) .and $\mu_{o:n}^{(k)} = 0$ for convention .

Proof: If we replace r with (r – 1) in equation (1.1) we get

$$F_{r-1:n}(x) = \sum_{i=r-1}^{n} \frac{1}{i!(n-i)!} Per\begin{bmatrix} F(x) & 1-F(x) \\ i & n-i \end{bmatrix}$$

expanding the summation on the first term, then

$$F_{r-1:n}(x) = F_{r:n}(x) + \frac{1}{(r-1)!(n-r+1)!} Per\begin{bmatrix} F(x) & 1-F(x) \\ r-1 & n-r+1 \end{bmatrix}$$

which is equivalent to

$$F_{r-1:n}(x) = F_{r:n}(x) + \sum_{p} \prod_{j=1}^{r-1} F_{ij}(x) \prod_{j=r}^{n} \left(1 - F_{i_{n-j+1}}(x)\right) \quad (17)$$

where the summation P extends over all permutation $(i_1, i_2, \ldots., i_n)$ of $(1, 2, \ldots\ldots, n)$ for which $1 \le i_1 \langle i_2 \langle \ldots. \langle i_{n-1} \le n$ and $1 \le i_r \langle i_{r+1} \langle \ldots.. \langle i_{r-1} \le n$. Now since $x_{i0} = \inf\{x: F_i(x) \rangle 0\} \ge 0$, for all i , then

$$\mu_{r:n}^{(k)} = E\left(X_{r:n}^{(k)}\right) = k\int_0^{\infty} x^{k-1}\left(1 - F_{r:n}(x)\right) dx \quad (18)$$

substituting (17) in (18)

$$\mu_{r:n}^{(k)} = k\int_0^{\infty} x^{k-1}\left[\begin{array}{l} 1 - F_{r-1:n}(x) + \sum_{p} \prod_{j=1}^{r-1} F_{i_j}(x). \\ . \prod_{j=r}^{n}\left(1 - F_{i_{n-j+1}}(x)\right) dx \end{array}\right]$$

$$= k\int_0^{\infty} x^{k-1}\left(1 - F_{r-1:n}(x)\right) dx +$$

$$+ k\sum_{p} \int_0^{\infty} x^{k-1} \prod_{j=1}^{r-1} F_{ij}(x) \prod_{j=r}^{n}\left(1 - F_{i_{n-j+1}}(x)\right) dx$$

$$\therefore \mu_{r:n}^{(k)} = \mu_{r-1:n}^{(k)} + J_{r:n}^{(k)}$$

where

$$J_{r:n}^{(k)} = k\sum_{p} \int_0^{\infty} x^{k-1} \prod_{j=1}^{r-1} F_{ij}(x) \prod_{j=r}^{n}\left(1 - F_{i_{n-j+1}}(x)\right) dx \quad (19)$$

Now consider

$F_i(x) = 1-\left(1+x^{c}\right)^{-m_i}$, $x \ge 0$, it follows

$$\prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=r}^{n}\left(1-F_{i_{n-j+1}}(x)\right) = \prod_{j=1}^{r-1}\left[1-\left(1+x^c\right)^{-m_{i_j}}\right] \prod_{j=r}^{n}\left(1+x^c\right)^{-m_{i_j}}$$

$$= \prod_{j=1}^{r-1}\left[1-\left(1+x^c\right)^{-m_i}\right]\left(1+x^c\right)^{-\sum_{j=r}^{n} m_{i_j}}$$

$$=\left(1+x^c\right)^{-\sum_{j=r}^{n} m_{i_j}} \left[\begin{array}{l} 1-\prod_{j_1=1}^{r-1}\left(1+x^c\right)^{-m_{j_1}} + \sum_{1\le j_1 \langle j_2 \le r-1}\left(1+x^c\right)^{-\left(m_{i_1}+m_{i_2}\right)} \\ -\sum_{1\le j_1\langle j_2 \langle j_3 \le r-1}\left(1+x^c\right)^{-\left(m_{i_{j_1}}+m_{i_{j_2}}+m_{i_{j_3}}\right)} \\ +\ldots+(-1)^{r-1}\left(1+x^c\right)^{-\sum_{j=1}^{r-1} m_j} \end{array}\right]$$

$$\prod_{j=1}^{r-1} F_{i_j}(x) \prod_{j=r}^{n}\left(1-F_{i_{n-j+1}}(x)\right) = \left(1+x^c\right)^{-\sum_{j=r}^{n} m_{i_j}} - \sum_{j_1=1}^{r-1}\left(1+x^c\right)^{-\left(m_{i_{j_1}}+\sum_{j=r}^{n} m_{i_j}\right)}$$

$$+\sum_{1\le j_1 \langle j_2 \le r-1}\left(1+x^c\right)^{-\left[\left(m_{i_{j_1}}+m_{i_{j_2}}\right)+\sum_{j=r}^{n} m_{i_j}\right]}$$

$$-\sum_{1\le j_1 \langle j_2 \langle j_3 \le r-1}\left(1+x^c\right)^{-\left[\left(m_{i_{j_1}}+m_{i_{j_2}}+m_{i_{j_3}}\right)+\sum_{j=r}^{n} m_{i_j}\right]}$$

$$+\ldots\ldots\ldots\ldots$$

$$+(-1)^{r-2}\sum_{1\le j_1\langle j_2\langle j_3 \langle\ldots\langle j_{r-2} \le r-1}\left(1+x^c\right)^{-\left[\left(m_{i_{j_1}}+m_{i_{j_2}}+\ldots+m_{i_{j_{r-2}}}\right)+\sum_{j=r}^{n} m_{i_j}\right]}$$

$$+(-1)^{r-1}\left(1+x^c\right)^{-\left(\sum_{j=1}^{n-1} m_{i_j}\right)}, \qquad (20)$$

Substituting (20) in (19) and after simple calculation we get

$$J_{r:n}^{(k)} = k\sum_{p}\left[\begin{array}{l} \int_0^\infty x^{k-1}\left(1+x^c\right)^{-\sum_{j=r}^{n} m_{i_j}} dx \\ -\sum_{j_1=1}^{r-1}\int_0^\infty x^{k-1}\left(1+x^c\right)^{\left(m_{i_{j_1}}+\sum_{j=r}^{n} m_{i_j}\right)} dx \\ +\sum_{1\le j_1\langle j_2 \le r-1}\int_0^\infty x^{k-1} \\ \left(1+x^c\right)^{-\left[\left(m_{ij_1}+m_{ij_2}\right)+\sum_{j=r}^{n} m_{i_j}\right]} dx \\ -\sum_{1\le j_1\langle j_2\langle j_3 \le r-1}\int_0^\infty x^{k-1} \\ \left(1+x^c\right)^{-\left[\left(m_{ij_1}+m_{ij_2}+m_{ij_3}\right)+\sum_{j=r}^{n} m_{i_j}\right]} dx \\ +\ldots+(-1)^{r-2}\sum_{1\le j_1\langle j_2\langle\ldots\langle j_{r-2}\le r-1}\int_0^\infty x^{k-1} \\ \left(1+x^c\right)^{-\left[m_{i_{j_1}}+m_{ij_2}+\ldots+m_{i_{j_{r-2}}}+\sum_{j=r}^{n} m_{i_j}\right]} dx \\ +(-1)^{r-1}\int_0^\infty x^{k-1}\left(1+x^c\right)^{-\sum_{j=1}^{n} m_{i_j}} dx \end{array}\right]$$

Upon using the integration (13) we get

$$J_{r:n}^{(k)} = \frac{k}{c}\sum_{p}\left[B\sum_{j=r}^{n} m_{ij}-\frac{k}{c},\frac{k}{c}\right]$$
$$-\sum_{j}^{r-1} B\left[\left(m_{ij_1}+\sum_{j=r}^{n} m_{ij}\right)-\frac{k}{c},\frac{k}{c}\right]$$
$$+\sum_{1\le j_1\langle j_2\le r-1} B\left[\left(m_{i_{j_1}}+m_{i_{j_2}}\right)+\sum_{j=r}^{n} m_{ij}-\frac{k}{c},\frac{k}{c}\right]$$
$$-\sum_{1\le j_1\langle j_2\langle j_3\le r-1} B\left[\left(m_{i_{j_1}}+m_{i_{j_2}}+m_{i_{j_3}}\right)+\sum_{j=r}^{n} m_{ij}-\frac{k}{c},\frac{k}{c}\right]$$
$$+\ldots.. +(-1)^{r-2}\sum_{1\le j_1\langle j_2\langle j_3\langle\ldots\langle j_{r-2}\le r-1}$$
$$B\left[\left(m_{i_{j_1}}+m_{i_{j_2}}+\ldots+m_{i_{j_{r-2}}}\right)+\sum_{j=r}^{n} m_{ij}-\frac{k}{c},\frac{k}{c}\right]$$
$$+(-1)^{r-1}\beta\left[\sum_{j=1}^{n} m_{ij}-\frac{k}{c},\frac{k}{c}\right] \qquad (21)$$

By using the fact that $\sum_{p}(1) = \binom{n}{r-1}$ and

$$\sum_{1\le j_1 \langle j_2 \langle j_3 \langle \ldots \le j_m \le n} \binom{n}{m} \text{ for all } n \ge m$$

we get

$$J_{r:n}^{(k)} = \sum_{j=1}^{n}(-1)^{j-1}\frac{k}{c}\, a_j\, \mathrm{I}_{n-r+j} \qquad (22)$$

where the sequence $\left\{\mathrm{I}_j\right\}_{j=n-r+1}^{j=n}$ is defined in (5) and $a_j = \frac{(n-r+j)!}{(n-r+1)!(j-1)!}$

since

$$\binom{n}{r-1}\binom{r-1}{j-1} = a_j\binom{n}{r-1}$$

which completes the proof.

CONCLUSION

By recursively applying equation (5) starting with the maximum $\mu_{n:n}^{(k)}$ in (4) one can deduct all moments of all order statistics $\mu_{r:n}^{(k)}$, $r \le n$ from Burr type XII distributions. One only needs to compute the sequence $\left\{I_j\right\}_{j=1}^{n}$ which is given by (5). This sequence is very simple to evaluate . For example if n=3, we get

$\mu_{3:3}^{(k)} = \frac{k}{c}(I_1 - I_2 + I_3)$

where

$I_1 = B(m_1 - \frac{k}{c}, \frac{k}{c}) + B(m_2 - \frac{k}{c}, \frac{k}{c}) + B(m_3 - \frac{k}{c}, \frac{k}{c})$

$I_2 = B(m_1 + m_2 - \frac{k}{c}, \frac{k}{c}) + B(m_1 + m_3 - \frac{k}{c}, \frac{k}{c}) + B(m_2 + m_3 - \frac{k}{c}, \frac{k}{c})$

$I_3 = B(m_1 + m_2 + m_3 - \frac{k}{c}, \frac{k}{c})$

$\mu_{1:3}^{(k)} = \frac{k}{c} I_3$

$\mu_{2:3}^{(k)} = \frac{k}{c}[I_2 - 3I_3]$

These results can be put in the following table

The moments $\mu_{r:n}^{(k)}, r \le n$ of order statistics arising from non-identically Burr type XII random variables with n=3

$\mu_{3:3}^{*(K)}$	I_1	$-I_2$	$+I_3$
$\mu_{2:3}^{*(K)}$		I_2	$-2I_3$
$\mu_{1:3}^{*(K)}$			I_3

$\mu_{r:n}^{*(k)} = \frac{c}{k}\mu_{r:n}^{(k)}$. Generalization of this table is mentioned in [13] .

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