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Quarter-Sweep Projected Modified Gauss-Seidel Algorithm Applied to Linear Complementarity Problem

[1]W.S. Koh, [1]J. Sulaiman and [2]R. Mail
[1]Mathematics with Economics Programme, School of Science and Technology,
[2]Accountancy Programme, School of Business and Economics,
University Malaysia Sabah, Locked Bag 2073, 88999 Kota Kinabalu, Sabah, Malaysia

Abstract: Problem statement: Modified Gauss-Seidel (MGS) was developed in order to improve the convergence rate of classical iterative method in solving linear system. In solving linear system iteratively, it takes longer time when many computational points involved. It is known that by applying quarter-sweep iteration scheme, it can decrease the computational operations without altering the accuracy. In this study, we investigated the effectiveness of the new Quarter-Sweep Projected Modified Gauss-Seidel (QSPMGS) iterative method in solving a Linear Complementarity Problem (LCP). **Approach:** The LCP we looked into is the LCP arise in American option pricing problem. Actually, American option is a Partial Differential Complementarity Problem (PDCP). By using full-, half- and quarter-sweep Crank-Nicolson finite difference schemes, the problem was reduced to Linear Complementarity Problem (LCP). **Results:** Several numerical experiments were carried out to test the effectiveness of QSPMGS method in terms of number of iterations, computational time and Root Mean Square Error (RMSE). Comparisons were made with full-, half- and quarter-sweep algorithm based on Projected Gauss-Seidel (PGS) and Projected Modified Gauss-Seidel (PMGS) methods. Thus, the experimental results showed that the QSPMGS iterative method has the least number of iterations and shortest computational time. The RMSE of all tested methods are in good agreement. **Conclusion:** QSPMGS is the most effective among the tested iterative methods in solving LCP whereby it is fastest and the accuracy remains the same.

Key words: Projected modified gauss-seidel, quarter-sweep iteration, linear complementarity problem, Crank-Nicolson scheme

INTRODUCTION

The Linear Complementarity Problem (LCP) is normally applied in the area of computational mechanics, financial engineering and other disciplines in engineering, science and economics. The widely applications of LCP are because it corresponds to the notion of equilibrium and constraint optimization problems (Ferris and Pang, 1997).

In order to define the LCP, consider a matrix M, vector q and unknown vector z. Then, the unknown vector z will be solved in the following conditions:

$$z \geq 0 \tag{1}$$

$$Mz \geq q \tag{2}$$

$$z(Mz - q) = 0 \tag{3}$$

The above formulations are the standard LCP. In this study, we will look into an implicit type of LCP whereby there exists another function y which plays an important role (Koulisianis and Papatheodorou, 2003):

$$z \geq y \tag{4}$$

$$Mz \geq q \tag{5}$$

$$(z - y)(Mz - q) = 0 \tag{6}$$

Actually, we can solve the LCP by using either direct or iterative methods. However, when we deal with a large sparse linear system, iterative method is preferable. Moreover, it does not consume much memory compared to direct method.

The aim of this study is to introduce a new iterative method known as Quarter-Sweep Projected Modified Gauss-Seidel (QSPMGS) algorithm which will

Corresponding Author: W.S. Koh, Mathematics with Economics Programme, School of Science and Technology, University Malaysia Sabah, Locked Bag 2073, 88999 Kota Kinabalu, Sabah, Malaysia

accelerate the convergence rate. It is the combination of Quarter-Sweep approximation scheme with Projected Modified Gauss-Seidel (PMGS) algorithm. Quarter-sweep iteration scheme is known to be effective to reduce the computational operations and thus speeds up the computational time without altering the accuracy; Sulaiman *et al*. (2004; 2009). The PMGS algorithm is a preconditioned iterative method based on the Modified Gauss-Seidel (MGS) method, established by Gunawardena *et al.* (1991) for the purpose of improving the convergence rate of classical iterative methods. Since then, many studies about the MGS method have been carried out like Li and Sun (2000) and Li (2005). Actually, Quarter-Sweep Modified Gauss-Seidel (QSMGS) had been applied to solve PDE in European option pricing problem, Koh and Sulaiman (2009). For verification of the new QSPMGS algorithm in solving LCP, we will examine it in the case of American option pricing.

As American option pricing model involves Partial Differential Complementarity Problem, (PDCP), Crank-Nicolson (CN) scheme will be applied to discretize the PDCP into a LCP. Full-, half- and quarter-sweep CN schemes for approximation of the PDCP will be presented. Then, we will show how the generated LCP solved by PMGS method. Several numerical experiments will be carried out in a family of PGS methods, namely Full-Sweep Projected Gauss-Seidel (FSPGS), Half-Sweep Projected Gauss-Seidel (HSPGS), Quarter-Sweep Projected Gauss-Seidel (QSPGS), Full-Sweep Projected Modified Gauss-Seidel (FSPMGS), Half-Sweep Projected Modified Gauss-Seidel (HSPMGS) and Quarter-Sweep Projected Modified Gauss-Seidel (QSPMGS) methods.

Case study: American option pricing model: Option is a financial instrument which allows the holder to trade a certain asset in future time with a certain price. The two major styles of options are European and American options. Generally, the difference between them is in the trading aspect as European option can only be traded at the expiration time while American option can be traded at any time before or on the maturity time. Due to this reason, the pricing of American option involved PDCP. The PDCP is shown as follows (Tavella and Randall, 2000):

$$v \geq g \tag{7}$$

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial v^2}{\partial s^2} + rs\frac{\partial v}{\partial s} \leq rv \tag{8}$$

$$\left(\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial v^2}{\partial s^2} + rs\frac{\partial v}{\partial s} - rv\right)(v - g) = 0 \tag{9}$$

Where:
v = Value of the options
t = Time
s = Underlying asset price
σ = Volatility of the asset price
r = Risk free interest rate
g = Payoff function of the option

The final time condition can be defined as follows (Ikonen and Toivanen, 2007):

$$v(s,T) = g = \begin{cases} \max(s(T) - K, 0) & \text{for call option} \\ \max(K - s(T), 0) & \text{for put option} \end{cases} \tag{10}$$

Where:
K = The exercise price
T = Maturity time

The boundary conditions for the American option will be as (Ikonen and Toivanen, 2007):

$$v(0,t) \text{ and } v(S,t) = S - K \tag{11}$$

$$v(0,t) = K \text{ and } v(S,t) = 0 \tag{12}$$

where, S is the maximum asset price whereby it is sufficiently large. The boundary conditions given in (11) and (12) correspond to American call and put options respectively.

MATERIALS AND METHODS

Quarter-sweep Crank-Nicolson scheme: The finite grid network for the full-, half- and quarter-sweep approximation schemes are illustrated in Fig. 1. The solid node points shown in Fig. 1 are the points that will be considered by using full-, half- and quarter-sweep iterative methods. However, the values for the remaining points will be approximated by using direct method, Sulaiman *et al.* (2004; 2009) and Koh and Sulaiman (2009). The PDE in (9) that is Black-Scholes PDE (Black and Scholes, 1973) can be discretized as follows (Tavella and Randall, 2000; Koh and Sulaiman, 2009):

$$\begin{aligned} \frac{v_{i,j+1} - v_{i,j}}{\Delta t} &= \\ &-\sigma^2 (s_0 + ip\Delta s)^2 \left(\frac{\begin{matrix} v_{i-p,j} - 2v_{i,j} + v_{i+p,j} + \\ v_{i-p,j+1} - 2v_{i,j+1} + v_{i+p,j+1} \end{matrix}}{4(p\Delta s)^2} \right) \\ &-r(s_0 + ip\Delta s)\left(\frac{\begin{matrix} v_{i+p,j} - v_{i-p,j} + \\ v_{i+p,j+1} - v_{i-p,j+1} \end{matrix}}{4p\Delta s} \right) + r\left(\frac{v_{i,j} + v_{i,j+1}}{2}\right) \end{aligned} \tag{13}$$

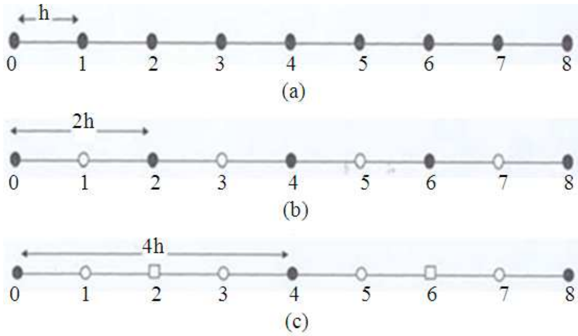

Fig. 1: (a-c) the node points for the full-, half- and quarter-sweep cases respectively

Then the approximation Eq. 13 can be simplified in the following equation:

$$c_i v_{i-p,j} + a_i v_{i,j} + b_i v_{i+p,j} = f_{i,j+1} \tag{14}$$

Where:

$$\beta_i = -\frac{1}{2}\sigma^2(s_0 + i\Delta s)$$

$$\lambda_i = -r(s_0 + i\Delta s)$$

$$\theta = \frac{1}{\Delta t}$$

$$c_i = \frac{1}{2p\Delta s}\left(\frac{\beta_i}{p\Delta s} - \frac{\lambda_i}{2}\right)$$

$$a_i = \theta + \frac{r}{2} - \frac{\beta_i}{(p\Delta s)^2}$$

$$b_i = \frac{1}{2p\Delta s}\left(\frac{\beta_i}{p\Delta s} + \frac{\lambda_i}{2}\right)$$

$$f_{i,j+1} = -cv_{i-p,j+1} + (2\theta - a)v_{i,j+1} - bv_{i+p,j+1}$$

If p is equal to 1, 2 or 4, it represents the full-, half-, or quarter-sweep schemes respectively. Then, we can rewrite (14) in a matrix form as:

$$A\underset{\sim}{v} = \underset{\sim}{f} \tag{15}$$

Where:

$$A = \begin{bmatrix} a_{1p} & b_{1p} & & & \\ c_{2p} & a_{2p} & b_{2p} & & \\ & \ddots & \ddots & \ddots & \\ & & & c_{np} & a_{np} \end{bmatrix}$$

$$\underset{\sim}{v} = \begin{bmatrix} v_{1,j} & v_{2,j} & \cdots & v_{n,j} \end{bmatrix}^T$$

$$\underset{\sim}{f} = \begin{bmatrix} f_{1,j+1} & f_{2,j+1} & \cdots & f_{n,j+1} \end{bmatrix}^T$$

A family of projected Gauss-Seidel iterative methods: As mentioned before, the Projected Modified Gauss-Seidel (PMGS) method is based on a preconditioned iterative method, namely Modified Gauss-Seidel (MGS) method (Gunawardena *et al*., 1991). In order to develop and implement a family of PGS algorithm, consider (15) and multiply both sides of the equation with preconditioned such as:

$$PA\underset{\sim}{v} = P\underset{\sim}{f} \tag{16}$$

Where:

$$P = I + S$$

$$S = \begin{bmatrix} 0 & -\alpha b_{1p} & 0 & \cdots & 0 \\ 0 & 0 & -\alpha b_{2p} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha b_{n-1p} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

I = Identity matrix

When α = 0, it is the classical Gauss-Seidel (GS) iteration, while if α = 1, it will become MGS method (Gunawardena *et al*., 1991; Koh and Sulaiman, 2009). Based on (16), the linear system can be rewritten as:

$$A^*\underset{\sim}{v} = \underset{\sim}{f}^* \tag{17}$$

Where:

$$A^* = PA$$

$$f^* = Pf$$

By using the linear system generated in (17), the PDCP in (9) can be shown as:

$$\left(A^*\underset{\sim}{v} - \underset{\sim}{f}^*\right)(v - g) = 0 \tag{18}$$

Now, a LCP has been generated from the PDCP and has the similar form as LCP in (4-6). By considering (18) and the PDCP defined in (7-9), the algorithms of the family of PGS methods will be generally described in Algorithm 1:

Algorithm 1:

i. Initializing all the parameters. Set k = 0.
ii. General iteration

$$x_i^{k+1} = \frac{1}{A*_{ii}}\left(f*_i - \sum_{j=1}^{i-p} A*_{ij}\, v_j^{k+1} - \sum_{j=i+p}^{i-n} A*_{ij}\, v_j^{k} \right)$$

If $x_i^{k+1} < g$ then $v_i^{k+1} = g$

Else $v_i^{k+1} = x_i^{k+1}$

iii. Convergence test.
If the error tolerance is satisfied, the value option at that time is v_i^{k+1} and the algorithm end.
Else, set k = k+1 and go to Step ii.

RESULTS

Several numerical experiments will be performed to examine the effectiveness of FSPGS, HSPGS, QSPGS, FSPMGS, HSPMGS and QSPMGS. The criteria concerned in these experiments include the number of iterations, computational time and Root Mean Square Error (RMSE). The parameters used in these experiments are taken from Hon (2002) whereby K = 100, r = 0.1, σ = 0.30, T = 1(year), $s \in [e^{-5}, e^{7}]$. The matrix sizes tested are 512, 1024, 2048, 4096, 8192 and 16384. As for the time steps, we will have 100 time steps which means Δt will be 0.01. The error tolerance $\varepsilon = 10^{-10}$ is selected for the convergence test. For comparison, the numerical results obtained will be compared with the results of Binomial method (Hon, 2002). Table 1 presents the experimental results. The results are also illustrated in Fig. 2 and 3.

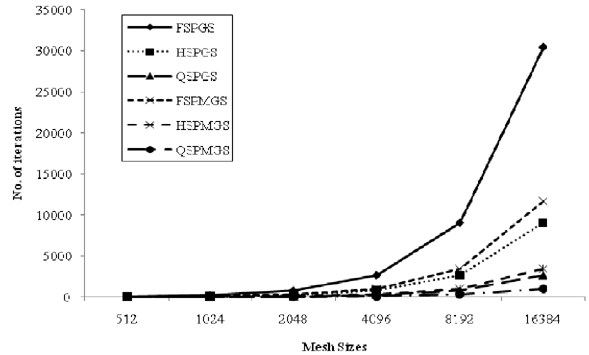

Fig. 2: Number of iterations versus mesh sizes of the FSGS FSPGS, HSPGS, QSPGS, FSPMGS, HSPMGS and QSPMGS methods

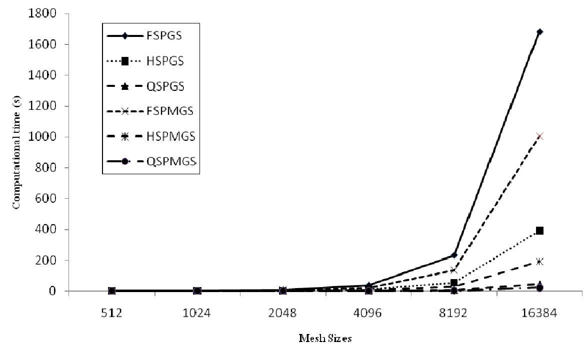

Fig. 3: Computational time (sec) versus mesh sizes of the FSGS FSPGS, HSPGS, QSPGS, FSPMGS, HSPMGS and QSPMGS methods

Table 1: Number of iterations, computational time and RMSE for FSPGS, HSPGS, QSPGS, FSPMGS, HSPMGS and QSPMGS methods

	Mesh size					
Method	512	1024	2048	4096	8192	16384
Number of iterations						
FSPGS	69	230	785	2673	9053	30,432
HSPGS	23	69	230	785	2673	9053
QSPGS	10	23	69	230	785	2673
FSPMGS	28	88	296	1012	3445	11651
HSPMGS	11	28	88	296	1012	3445
QSPMGS	6	11	28	88	296	1012
Computational time (s)						
FSPGS	0.11	0.79	4.67	33.69	230.72	1679.72
HSPGS	0.06	0.19	1.14	7.78	53.15	389.02
QSPGS	0.01	0.03	0.14	0.87	5.53	44.95
FSPMGS	0.08	0.39	2.51	18.31	135.25	1002.56
HSPMGS	0.01	0.09	0.56	3.54	25.21	190.21
QSPMGS	0.00	0.02	0.07	0.47	2.85	21.09
RMSE						
FSPGS	0.016794	0.016906	0.017116	0.017247	0.020091	0.021419
HSPGS	0.022725	0.016794	0.016906	0.017116	0.017247	0.020091
QSPGS	0.094417	0.022725	0.016794	0.016906	0.017116	0.017247
FSPMGS	0.016794	0.016906	0.017116	0.017247	0.020091	0.021419
HSPMGS	0.022725	0.016794	0.016906	0.017116	0.017247	0.020091
QSPMGS	0.094417	0.022725	0.016794	0.016906	0.017116	0.017247

DISCUSSION

In the computational experiments, we have tested the iterative methods with different mesh sizes in terms of number of iterations, computational time and RMSE.

As the mesh sizes go larger, the more points could be considered, which means option with more precise underlying assets price could be priced. Based on the results for different mesh sizes, the accuracies of all iterative methods are in good agreement. This means that half- and quarter-sweep algorithms computed only parts of the entire node points and their accuracies don't alter.

According to Fig. 2 and 3, QSPMGS has the lowest computational time as while as the least number of iterations. Through numerical results in Table 1, percentage reduction for number of iterations of HSPGS, QSPGS, FSPMGS, HSPMGS and QSPMGS are about 66.67-70.70, 85.51-91.40, 59.42-62.29, 84.06-88.92 and 91.30-96.73% respectively compare to FSPGS. In terms of execution time, HSPGS, QSPGS, FSPMGS, HSPMGS and QSPMGS algorithms are faster approximately 45.45-76.96, 90.91-97.60, 27.27-50.63, 88.01-90.91 and 98.76-100% than FSPGS algorithm. As we can see in Table 1, the QSPMGS takes only 21.09 seconds for largest mesh size, 16384.

CONCLUSION

In this study, the effectiveness of the Quarter-Sweep Projected Modified Gauss-Seidel (QSPMGS) algorithm has been examined in solving Linear Complementarity Problem (LCP). In the experiments involved full, half- and quarter-sweep algorithm based on Projected Gauss-Seidel (PGS) and Projected Modified Gauss-Seidel (PMGS) methods, QSPMGS proved to be the most effective iterative method. QSPMGS converges faster by having the least number of iterations and thus speed up the execution time.

For future work, further investigation for the capability of the combination of quarter-sweep iteration with MGS method needs to be performed for solving various multi-dimensional problems (Ibrahim and Abdullah, 1995; Tavella and Randall, 2000; Sulaiman *et al.*, 2009). In fact, we can consider improving the proposed method by implementing block iterative approach.

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