



### Control of Hybrid Systems With Independent Continuous and Discrete Goals

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**Abstract:** This paper deals with the control of a class of perfectly modelled linear hybrid systems which consist of two, in general coupled, subsystems one being continuous-time while the second one is digital. Both subsystems are driven by continuous time and sampled values of a control input. The description of the overall system is given through an extended hybrid system, which is purely discrete at sampling instants, which has two, in general, coupled continuous-time and discrete-time substates. The discrete-time substate is jointly defined by the digital substate and the samples of the continuous substate while being driven by the sampled input. The control objective has a double nature and it consists of the achievement of separate continuous-time and discrete-time model-matching objectives with respect to two predefined stable reference models. It is achieved by synthesizing a dynamic hybrid controller consisting of a continuous subcontroller and a discrete one. Each of those controllers has its own control objective.

**KeyWords:** Hybrid Systems, Control of Hybrid Systems

#### INTRODUCTION

Hybrid systems have received important attention in the last years (see, for instance, <sup>[2,3]</sup>). In particular, the optimization of inputs and the fundamental properties of such systems have received attention in <sup>[2]</sup> and the multirate sampling of such systems has been studied in <sup>[10,3]</sup>. The importance of those systems arises from the fact that continuous and digital subsystems usually operate in a combined and integrated fashion. Another important reason to deal with such systems is that it becomes sometimes suitable the use of either discrete-time or digital controllers for continuous plants by technological implementability reasons <sup>[8,9]</sup>. In this paper, a wide class of linear hybrid systems proposed in <sup>[2]</sup> and also dealt with in <sup>[3]</sup> is considered in the context of model-matching designs. Such systems are characterized by the continuous substate being forced by both the current input in continuous time and its sampled value at the last preceding sampling instant as well. The objective of this paper is the design of an hybrid controller that allows the hybrid plant to achieve, in

general, separate continuous-time and discrete-time model-following objectives in the perfectly modelled situation. In this way, the continuous-time and discrete-time closed-loop dynamics can be separately designed through the synthesis of two subcontrollers which give together the overall, in general, hybrid controller. The subcontroller designed for accomplishing with the discrete-time control objective has a discrete-time nature while that designed to accomplish with the continuous-time objective is of a mixed continuous-time and discrete-time nature. Several particular cases which are included in the general framework are for instance:

- The choice of only a continuous-time reference model. Thus, its digital transfer function is used as discrete model for controller synthesis at sampling instants
- The use of only a discrete-time reference model under a piecewise constant plant input in between sampling instants. In such a case, the overall scheme becomes a discrete-time one
- The use of the discrete-time reference model for periodic testing of the current closed-loop

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performance designed for a continuous-time reference dynamics . If the test fails then the continuous-time objective can be on - line modified in terms of re-adjustment of the input to the (continuous-time) reference model or high-frequency gain re-adjustment to modify either the transient reference signal or the steady-state reference set point.

Each subcontroller is designed for the achievement of the corresponding model-following objective in the absence of plant unmodelled dynamics. Also, as a part of the design , each subcontroller generates a compensating signal to annihilate the coupling signals generated from the continuous signals to the discretized output , for the discrete-time control objective , or viceversa. when dealing with the continuous control objective. Such coupling signals are inherent to the structure of the open-loop hybrid plant . Finally, the overall controller is robust against a class of unmodelled dynamics and uniformly bounded state and measurement noises.

**Hybrid plant description:** The hybrid plant Consider the next single-input single-output hybrid linear system ( Kabamba and Hara 1994; De la Sen 1996):

$$\begin{aligned} \dot{x}_c(t) &= A_c x_c(t) + A_{cs} x_c[k] + A_{cd} x_d[k] \\ &+ b_c u(t) + b_{cs} u[k] \\ x_d[k+1] &= A_d x_d[k] + A_{ds} x_c[k] + b_d u(t) + b_{ds} u[k] \\ y(t) &= c_c^T x_c(t) + c_{cs}^T x_c[k] + c_d^T x_d[k] + d_c u(t) + d_d u[k] \end{aligned} \quad (1)$$

for  $t \in [kT, (k+1)T)$ ; all nonnegative integer  $k$ , with  $T$  being the sampling period, where  $x_c(\cdot)$  and  $x_d[\cdot]$  are, respectively, the  $n_c$  and  $n_d$  continuous and digital subvectors and  $u(\cdot)$  and  $y(\cdot)$  are the scalar input and output. The continuous time argument is denoted by ' $t$ ' while the discrete time argument is denoted by ' $[k]$ ' and the associated continuous and digital variables are denoted correspondingly. Thus, a continuous variable at sampling instants is denoted in the same way as a digital variable so that  $x_c[k] = x_c(kT)$  and  $u[k] = u(kT)$  in (1). In that way, there is no distinction in the treatment of digital and time-discretized variables. The orders of all the real constant matrices in (1) agree with the dimensions of the substates and scalar input and output.

**Description of (1) at sampling instants :** The input / output solution of (1) at sampling instants is given by the ARMA- model :

$$Q_d(q) y[k] = P_d(q) u[k] + Q_d(q) (c^T \omega[k]) \quad (2)$$

for all nonnegative integer  $k$ , where  $c^T = c_c^T + c_{cs}^T$ ,  $Q_d(q)$  and  $P_d(q)$  are polynomials of real coefficients defined in Appendix A ( see eqns. (A.4) and (A.6) ) of degree  $n = n_c + n_d$  and  $q$  is the one-step -ahead shift operator. The ARMA - model (2) is obtained from the extended discrete-time system of state  $x[k] = [x_c^T[k], x_d^T[k]]^T$  obtained from (1).

**Description of (1) inbetween sampling instants:**

The input / output differential-difference relationship for (1) inbetween sampling instants is given by

$$Q_c(D) Q_d(q) y(t) = P_c(D) Q_d(q) u(t) + Q_c(D) \{ N_{cd}^u(D,q) u[k] + N_{cd}^{\omega T}(D,q) w[k] \} \quad (3)$$

for  $t \in [kT, (k+1)T)$ ; all integer  $k \geq 0$ , with  $q$  and  $D$  being the one-step ahead time -shift and time-derivative defined by  $q v(t) = v(t+T)$  and  $\dot{v}(t) = Dv(t)$ , respectively, for any differentiable signal  $v(t)$  in the continuous-time argument  $t$ , where  $Q_d(q)$  and  $P_d(q)$  are the polynomials in (2) while  $Q_c(q)$  and  $P_c(q)$  are polynomials of degree  $n_c$  and  $N_{cd}^u$  and  $N_{cd}^{\omega T}$  are a scalar polynomial and a two-variable  $n_c$  - polynomial matrix which have been obtained from the above parameters but the parametrical definition and its development are omitted by space reasons. Note that the term in brackets in the right - hand- side of (3) is a coupling signal from the digital substate and discretized input to the continuous subsystem of (1). The description (3) is obtained from an extended hybrid system of continuous- time substate  $x_c(t)$  and the discrete-time substate  $x[k] = [x_c^T[k], x_d^T[k]]^T$  used for obtaining (2) at sampling instants . The next simple descriptive example illustrates the decomposition in continuous / discrete ( or digital ) state variables of an input / output linear mapping involving the operators  $D$  and  $q$  as it occurs in the general description of (3).

**Example:** Consider the input/ output linear mapping  $v(t) = H_1(D) H_3(q) \delta[k] + H_2(D) v(t)$  driven by the discrete input  $\delta[k]$  and the continuous one  $v(t)$  where  $H_1(D) = \frac{D+a}{D+b}$ ;  $H_2(D) = \frac{1}{D+c}$ ;  $H_3(q) = \frac{q+1}{q+2}$  with  $a$ ,  $b$  and  $c$  being real constants.

Define now two continuous-time variables  $v_1(t)$  and  $v_2(t)$  and a digital variable  $\delta_1[k]$  given by the dynamics  $v_1(t) = \frac{D+a}{D+b} \delta_1[k]$ ,  $v_2(t) = \frac{1}{D+c} v(t)$  and  $v_3[k] = \frac{q+1}{q+2} \delta[k]$ . Thus, the overall state-space representation is described by

$$\begin{aligned} v(t) &= v_1(t) + v_2(t) + v_3[k] \\ \dot{v}_1(t) &= -bv_1(t) + (a-b)v_3[k] \\ v_1(t) &= v_1(t) + v_3[k] \\ \dot{v}_2(t) &= -cv_2(t) + v(t); \\ v_3[k] &= -2v_3[k-1] + \delta[k] + \delta[k-1] \end{aligned}$$

subject to initial conditions  $v_i(0) = v_{i0}$  ( $i = 1, 2$ ) and  $v_3[0] = v_{30}$ .  $\square$

**Remark 1:** The description of (3) also describes eqns. 1 at sampling instants and results to be

$$\begin{aligned} Q_c(D)Q_d(q)y[k] &= [P_c(D)Q_d(D) \\ + Q_c(D)N_{cd}^u(D,q)]u[k] \\ + Q_c(D)N_{cd}^{\omega T}(D,q)\omega[k] \end{aligned}$$

whose discrete-time solution is (2). Note through a comparison with (3) that the parametrization of the differential-difference solution to (1) becomes modified at sampling instants with respect to the intersample parametrization since additive terms involving the sampled continuous substate and sampled input result from the plant parametrization given by (1) at sampling instants.

**Global exponential stability conditions for the open-loop plant:** The global exponential stability of the unforced system (1) is only dependent on the stability of the  $A$ -matrix defined by

$$\begin{bmatrix} e^{A_c T} & [I + (\int_0^T e^{-A_c \tau} d\tau) A_{cd}] \\ A_{ds} & e^{A_d T} (\int_0^T e^{-A_c \tau} d\tau) A_d \end{bmatrix} \quad (4)$$

obtained after omitted calculations. This follows from building the extended unforced discrete dynamics  $x[k+1] = A x[k]$  with  $x[k] = (x_c^T[k], x_d^T[k])^T$ . Thus, the continuous-time solution of the continuous substate in (1) satisfies:

$$x_c(kT + \tau) = [e^{A_c \tau} (I + \int_0^\tau e^{-A_c \tau'} d\tau') A_{cs}]$$

$$, e^{A_c \tau} (\int_0^\tau e^{-A_c \tau'} d\tau') A_{cd}] x[k]$$

. Thus, if  $A$  is strictly Hurwitzian, then  $x_d[k]$ ,  $x_c[k]$  and  $x_c(t)$  converge to zero exponentially fast exponentially fast for any bounded initial conditions. The next result, whose proof is omitted, is concerned with the stability of the  $A$ -matrix under that of  $A_c$  and  $A_d$  provided that the coupling signals between continuous and discretized (or, indistinctly, digital) variables are sufficiently small.

**Proposition 1:** Assume that  $A_c$  and  $A_d$  are strictly Hurwitzian with their maximum eigenvalues satisfying:

$$\begin{aligned} e^{-\rho T} &\leq \left| \lambda_{\max}(e^{A_c T}) \right| \\ &\leq e^{-\rho T} \text{ (i.e., } -\rho \leq \lambda_{\max}(A_c) \leq -\rho') \end{aligned}$$

and  $|\lambda_{\max}(A_d)| \leq e^{-\rho T}$ . Thus, the open-loop unforced plant is globally exponentially stable if  $\left| \lambda_{\max} \left( \begin{bmatrix} A_{cs} & A_d \\ A_{ds} & I \end{bmatrix} \right) \right| < \text{Min}(e^{\rho T} - 1, \frac{\rho'(e^{\rho T} - 1)}{e^{\rho T} - 1})$

**Controller synthesis:** General design philosophy and Assumptions. The controller to be synthesized will consist of two subcontrollers each one being designed to satisfy a different (respectively, continuous-time or discrete-time) control objective, namely:

**Objective 1:**  $u[k] = u[kT]$  is generated in such a way that a prescribed stable discrete reference model of transfer function  $W_{md}(q)$  is matched at sampling instants. A discrete subcontroller (Subcontroller 1) which will be then synthesized accomplishes with this control objective. As a part of the design, the coupling signal in (2) from the continuous-time subsystem to the discrete-time subsystem, caused by the signal  $\omega[k] = [e^{A_c T} (\int_0^T e^{-A_c t} u(kT+t) dt) b_c]$ , that

includes the contribution of the continuous-time input over one sampling period to the output at sampling instants, is annihilated by synthesizing the appropriate compensator as addressed below.

**Objective 2:**  $u(t)$  ( $t \neq kT$ ) is generated in such a way that the closed-loop system matches a prescribed stable continuous-time reference model of

transfer function  $W_{mc}(D)$  inbetween sampling instants. A mixed continuous / discrete subcontroller (Subcontroller 2) is synthesized to accomplish with such a control objective. As a part of the design, the couplings between the discretized signals  $u[k]$  and  $\omega[k]$  and the continuous subsystem are cancelled by synthesizing the appropriate compensator as addressed below.

Since  $u[k]$  and  $u(t)$ ,  $t \in [kT, (k+1)T]$ , all nonnegative integer  $k$  are, in general, synthesized to satisfy two different control objectives, discontinuities of the control input at sampling instants occur in general. Also, there are input discontinuities caused by the influence in the feedback signals of the modification of the digital substate at sampling instants while it is kept constant inbetween sampling instants. When suitable, the two reference models can be appropriately related to each other in order to state the problem with a unique control objective as discussed later. Those input discontinuities translate in output discontinuities at sampling instants in the more general case when  $W_{mc}(D)$  and  $W_{md}(q)$  are chosen independently. The combined objective can be intuitively figured as of the actions of Subcontrollers 1-2 synthesized to satisfy the Control Objectives 1-2. There are two control channels integrated in the actuator that generate the input 'at' and 'inbetween' sampling instants as  $u(t) = u'[k]$  ( $t = kT$ );  $u(t) = u'(t)$  ( $t \neq kT$ ) Channel 1 is used to generate (inbetween sampling instants) the input for model-matching of  $W_{md}(q)$  while Channel 2 is used to match  $W_{mc}(D)$ . Note that once Channel 1 modifies its state, it supplies  $u[k]$  at sampling instants.

**Assumptions:**

1.  $P_d(q)$  and  $P_c(D)$  have all their zeros in  $|q| < 1$  and  $\text{Re}(D) < 0$ .
2. All common zeros of  $P_d(q)$  and  $Q_d(q)$  (of  $P_c(D)$  and  $Q_c(D)$ ), if any, are strictly Hurwitzian and closed-loop zeros and poles of the discrete-time (continuous-time) dynamics, i.e., they are zero-pole cancellations of  $W_{md}(q)$  in  $|q| < 1$  (of  $W_{mc}(D)$  in  $\text{Re}(D) < 0$ ). Also, the zeros of  $P_c(D)$  and  $P_d(q)$  which are cancelled by the controller, if any, are closed-loop poles and thus poles of  $W_{mc}(D)$  and  $W_{md}(q)$ , respectively.

3.  $W_{mc}(D)$  and  $W_{md}(q)$  are proper, strictly Hurwitzian and of relative orders non less than those of  $P_c(D)/Q_c(D)$  and  $P_d(q)/Q_d(q)$ , respectively.  $\square$

Note that Assumption 1 means that both (open-loop) discrete and continuous-time descriptions eqns. 2 and 3 are inversely stable. Assumption 2 means that if any of the discrete or continuous plant dynamics is uncontrollable (i.e., there are zero-pole cancellations) then the associated uncontrollable modes have to be stable and closed-loop poles of the corresponding dynamics. The need for such an assumption will then arise from the solvability of the diophantine equations associated with the pole-placement problems of Objectives 1-2. Note also that if  $d = d_c + d_d$  ( $d_c$ ) is nonzero in (1) then  $P_d(q)/Q_d(q)$  ( $P_c(D)/Q_c(D)$ ) is nonstrictly proper and then the realizability of Subcontroller 1 (Subcontroller 2) is realizable for any realizable  $W_{md}(q)$  ( $W_{mc}(D)$ ). Thus, the relative order constraint of Assumption 3 holds automatically under the realizability of the discrete-time (continuous-time) reference model guaranteed by its properness of the first part of the assumption.

**Objective 1: Synthesis of Subcontroller 1 and Generation of  $u[k] = u(kT)$**

The next discrete control law is designed to achieve Objective 1 when the plant (1) is perfectly known and noisy-free:

$$u[k] = \frac{G_{1d}(q)}{L_d(q)} u[k] + \frac{G_{2d}(q)}{L_d(q)} y[k] + \frac{G_{3d}^T(q)}{L_d(q)} \omega[k] + \frac{R_{1d}(q)}{L_d(q)} r_{1d}[k] \quad (5)$$

The compensating signal  $r_{1d}[\cdot]$  is forwarded to the plant input from the reference model input  $r_d[k]$

$$\omega[k] = \left( \int_0^T e^{A_c(T-\tau)} u(kT+\tau) d\tau \right) b_c$$

according to generation laws given below. All the transfer functions in the above control law are expressed as quotients of polynomials and realizable. The above law is explicit as follows:

$$u[k] = C_{y_u}^d(q) y[k] + C_{\omega_u}^{dT}(q) \omega[k] + C_{r_u}^d(q) r_{1d}[k] \quad (6.a)$$

where the compensator transfer functions are

$$\begin{aligned} C_{yu}^d(q) &= \frac{G_{2d}(q)}{L_d(q) - G_{1d}(q)} \\ C_{\omega u}^d(q) &= \frac{G_{3d}(q)}{L_d(q) - G_{1d}(q)} \\ C_{r_1 u}^d(q) &= \frac{R_{1d}(q)}{L_d(q) - G_{1d}(q)L_d(q)} \end{aligned} \quad (6.b)$$

The problem of accomplishing with Objective 1 consists of designing the polynomials  $G_{id}(q)$  ( $i=1,2$ ) and  $R_{1d}(q)$ , the polynomial vector  $G_{3d}(q)$  as well as the compensating signal  $r_{1d}[k]$ , for a given stable  $L_d(q)$  so that  $W_{md}(q)$  is matched if the plant is perfectly known and free of unmodelled dynamics and noise. The next result addresses the controller design :

**Theorem 1:** Suppose that the control law (6) is applied,  $r_d[k]$  ( $k \geq 0$ ) is the uniformly bounded reference input sequence to  $W_{md}(q)$  and that the next assumptions hold :

4. Assumptions 1 - 3 hold for  $P_d(q)$ ,  $Q_d(q)$  and the poles of  $W_{md}(q)$ , and that all the roots of  $R_{1d}(q)$  and  $L_d(q)$  are in  $|q| < 1$ . Assume also that  $d_c = -d_d$  in (1) and  $\deg(R_{1d}) \leq \deg(L_d - G_{1d})$ .

5.  $P_d(q) = \bar{Q}_d(q)P'_d(q)$  and  $Q_d(q) = \bar{Q}_d(q)Q'_d(q)$  where  $\bar{Q}_d(q)$  is the strictly Hurwitzian (from Assumption 2) maximum common factor of  $P_d(q)$  and  $Q_d(q)$ . Also,  $P'_d(q) = P_{1d}(q)P_{2d}(q)$  with  $P_{1d}(q)$  being defined by the discrete strictly Hurwitzian plant zeros (from Assumption 2) which are not plant poles and they are transmitted to the reference model  $W_{md}(q) = B_{md}(q)/A_{md}(q)$ .

6.  $L_d(q)$  is factorized as  $L_d(q) = P_{2d}(q)L'_d(q)$  in (5).

Thus, the discrete closed-loop transfer function equalizes that of  $W_{md}(q)$  provided that Subcontroller 1 and its associated compensating signal  $r_{1d}[k]$  are synthesized as follows :

$$\begin{aligned} r_{1d}[k] &= \frac{B_{md}(q)}{R_{1d}(q)P_{2d}(q)} r_d[k] \\ &+ \frac{L_d(q) - G_{1d}(q)}{R_{1d}(q)} M_d c^T \omega[k] \end{aligned} \quad (7)$$

where  $G_{1d}(q) = G'_{1d}(q)P_{2d}(q)$ ,  $d = d_c + d_d$ ,  $c = c_c + c_{cs}$  with  $M_d(q)$  being an arbitrary polynomial satisfying  $\deg(M_d(q)) < \deg(L_d(q) - G_{1d}(q)) - \deg(R_{1d}(q))$ ,  $G_{3d}(q) = -(1 + R_{1d}M_d c)$ , and  $G'_{1d}(q)$ ,  $G_{2d}(q)$  being polynomials which are the unique solution to the diophantine equation :

$$\begin{aligned} Q'_d(q)G'_{1d}(q) + P_{1d}(q)G_{2d}(q) \\ = Q'_d(q)L'_d(q) - A''_{md}(q) \end{aligned} \quad (8)$$

subject to the degree constraints  $\deg(G_{2d}(q)) < \deg(Q'_d(q))$  or  $\deg(G'_{1d}(q))$

$< \deg(P_{1d}(q))$  for  $A''_{md}(q)$  being a polynomial satisfying the factorizations

$$\begin{aligned} A_{md}(q) &= \bar{Q}_d(q)A'_{md}(q) \\ &= \bar{Q}_d(q)P_{2d}(q)A''_{md}(q) \end{aligned} \quad (9)$$

which exist from Assumption 2.  $\square$

**Corollaries : 1.** Theorem 1 also holds under the same assumptions if  $G_{3d}$  is a rational function and the compensating signal in the controller satisfy :

$$G_{3d}(q) = \frac{Q'_d(q)(G'_{1d}(q) - L'_d(q))c}{P_{1d}(q)} \quad (10)$$

$$r_{1d}[k] = \frac{1}{R_{1d}(q)} \frac{B'_{md}(q)}{P_{2d}(q)} r_d[k] \quad (11)$$

and all the remaining compensators of the control law remaining identical as in Theorem 1.

2. Theorem 1 and Corollary 1 also apply directly to the regulation case with  $r_d[k] = r_{1d}[k] = 0$  with the closed-loop dynamics resulting to be  $A_{md}(q)y[k] = 0$ .  $\square$

The proof of Corollary 1 becomes direct from the application of Assumptions 3 - 6 of Theorem 1 and the use of the cancelled factors  $Q_d P_{2d}$  and

$G_{1d} = G'_{1d} P_{2d}$  to yield :



$$\begin{aligned}
 G_{3d} &= \frac{\bar{Q}_d Q'_d (G_{1d} - L_d) c}{P_d} \\
 &= \frac{Q'_d (G'_{1d} - L'_{1d}) c}{P_{1d}} \\
 \Rightarrow C_{r_{1u}}^d &= \frac{G_{3d}}{L'_d - G'_{1d}} = -\frac{Q'_d}{P'_d} = -\frac{Q_d}{P_d}
 \end{aligned}$$

which is nonstrictly proper and stable since  $P_d$  is strictly Hurwitzian and  $d_c \neq d_d$ . The use of the above relationships leads to

$$\begin{aligned}
 [(L_d - G_{1d})Q_d - P_d G_{2d}]y[k] \\
 = Q_d(q)R_{1d}(q)r_{1d}[k] \quad (12)
 \end{aligned}$$

from (7)-(11) and Corollaries 1-2 follow as Theorem 1. Corollary 2 follows when  $r_d[k] \equiv 0$ .  $\square$

Note that the main difference between the design of Theorem 1 and Corollary 1 is the choice of the compensator  $C_{r_{1d}}^d(q)$  in (6). In Theorem 1, this proper compensator of high-frequency being  $-d^{-1}c = -(d_c + d_d)^{-1}(c_c + c_{cs})$  which cancels the high-frequency gain of  $\frac{G'_{1d} - L'_d}{P_{1d}} Q'_d c$ .

Thus, the closed-loop dynamics depends on  $\omega[k-1]$  but not on  $\omega[k]$  and  $M_d(q)$  is kept arbitrary. However, the decomposition of all the transfer functions from the components of  $\omega[k]$  to  $u[k]$  in Corollary 1 with their high-frequency gains being cancelled is not used. The synthesis mechanism in that case is the choice of  $G_{3d}(q)$  such that the transfer function from  $\omega[k]$  to  $u[k]$  is cancelled.

**Objective 2: Synthesis of Subcontroller 2 and generation of  $u(t)$  ( $t \neq kT$ )**

The next control law is designed for the achievement of Objective 2 when the known plant is perfectly modelled and free-noise and has the following implicit structure:

$$\begin{aligned}
 u(t) &= \frac{G_{1c}(D,q)}{L_c(D,q)} u(t) + \frac{G_{2c}(D,q)}{L_c(D,q)} y(t) \\
 &+ \frac{G_{3c}(D,q)}{L_c(D,q)} u[k] + \frac{G_{4c}^T(D,q)}{L_c(D,q)} \omega[k] + \frac{R_{1c}(D)}{L_c(D,q)} r_{1c}(t) \quad (13)
 \end{aligned}$$

for all  $t \in (kT, (k+1)T)$  and all nonnegative integer  $k$ , with  $r_{1c}(t)$  being a compensating signal to be generated as a part of the controller design and  $L_c(D, q)$  being a strictly Hurwitzian two-variable polynomial. The various filters are formed by two variable polynomials and the associated hybrid realizations can be obtained as addressed in the given example. The above control law becomes explicated as follows:

$$\begin{aligned}
 u(t) &= C_{yu}^c(D,q)y(t) + C_{uu}^c(D,q)u[k] \\
 &+ C_{\omega u}^{cT}(D,q)\omega[k] + C_{r_{1u}}^c(D,q)r_{1c}[k] \quad (14)
 \end{aligned}$$

for all  $t \in (kT, (k+1)T)$  with

$$C_{yu}^c(D,q) = \frac{G_{2c}(D,q)}{L_c(D,q) - G_{1c}(D,q)} \quad (15)$$

$$C_{uu}^c(D,q) = \frac{G_{3c}(D,q)}{L_c(D,q) - G_{1c}(D,q)} \quad (16)$$

$$C_{\omega u}^c(D,q) = \frac{G_{4c}(D,q)}{L_c(D,q) - G_{1c}(D,q)} \quad (17)$$

$$C_{r_{1u}}^c(D,q) = \frac{R_{1c}(D)}{L_c(D,q) - G_{1c}(D,q)} \quad (18)$$

Note that the compensators of (15)-(18) are dependent on  $D$  and  $q$  because of structure of (3). The problem of fulfilling Objective 2 consists of synthesizing (14), subject to (15)-(18), as well as the compensating signal  $r_{1c}(\cdot)$  as addressed in the next result which applies the philosophy of Theorem 1 and Corollary 1 to the problem of model-matching of the continuous reference model. In the following, the degree of two-variable polynomials with respect to one of the variables is denoted with the corresponding subscript.

**Theorem 2:** Suppose that  $r_c(t)$  is the uniformly bounded reference input to  $W_{mc}(D)$  and that the next assumptions hold

7. Assumptions 1 - 3 hold for  $P_d(q)$ ,  $Q_d(q)$ ,  $P_c(D)$  and  $Q_c(D)$  and that the poles of  $W_{mc}(D)$  and all the roots of  $R_{1c}(D)$  and  $L_c(D)$  are in  $\text{Re}(D) < 0$ . Assume also that  $d_c \neq -d_d$ .

8.  $P_c(D)$  admits the polynomial factorization  $\bar{Q}_c(D)P_{1c}(D)P_{2c}(D)$  where  $\bar{Q}_c(D)$  includes the (stable) common roots of  $P_c(D)$  and  $Q_c(D)$ ,

$P_{1c}(D)$  contains eventual zeros of  $P_c(D)$  transmitted from the plant to the reference model and  $P_{2c}(D)$  includes the (stable) plant zeros which are closed-loop poles and controller poles.

Thus, the closed-loop dynamics is globally exponentially stable and defined by

$$A_{mc}(D)y(t) = B_{mc}(D)r_c(t) \quad (19)$$

if the compensators in (15)-(18) and compensating signal  $r_{1c}(t)$  are chosen to satisfy  $G_{1c}(D, q) = P_{2c}(D, q) G'_{1c}(D, q)$  where  $(G'_{1c}(D, q), G_{2c}(D, q))$  is a polynomial pair being a unique solution to the two-variable diophantine equation:

$$Q'_c(D)G'_{1c}(D, q) + P'_c(D)G_{2c}(D, q) = L'_c(D, q) - A''_{mc}(D, q) \quad (20)$$

with  $L'_c(D, q) = P_{2c}(D, q) L'_c(D, q)$  and  $A''_{mc}(D, q) = \overline{Q}_c(D, q)P_{2c}(D, q)A''_{mc}(D, q)$  subject to any of the two the next degree constraints

$$\begin{aligned} \deg_D(L'_c(D, q) - A''_{mc}(D, q)) &\leq \deg_D(G_{1c}(D, q)) \\ \deg_D(G_{2c}(D, q)) &< \deg(Q'_c(D)) = \deg(P'_c(D)) \end{aligned} \quad (21.a)$$

$$\begin{aligned} \deg_D(L'_c(D, q) - A''_{mc}(D, q)) &\leq \deg_D(G_{2c}(D, q)) \\ \deg_D(G_{1c}(D, q)) &< \deg(Q'_c(D)) = \deg(P'_c(D)) \end{aligned} \quad (21.b)$$

and

$$G_{3c}(D, q) = \frac{G'_{1c}(D, q) - L'_c(D, q)}{Q'_c(D)P_{1c}(D)Q_d(q)} N_{cd}^u(D, q) \quad (22.a)$$

$$G_{4c}(D, q) = \frac{G'_{1c}(D, q) - L'_c(D, q)}{Q'_c(D)P_{1c}(D)Q_d(q)} N_{cd}^\omega(D, q) \quad (22.b)$$

$$r_{1c}(t) = \frac{B'_{mc}(D)}{P_{2c}(D)R_{1c}(D)} r_c(t) \quad (22.c)$$

with  $B'_{mc}(D)$  being the free-design zeros of  $W_{mc}(D)$  (i.e., those of  $W_{mc}(D)$  excluding the factor  $\overline{Q}_c(D)P_{1c}(D)$ ). □

The proof is omitted by space reasons. Note that Theorem 2 applies the same philosophy for pole-placement for the continuous reference model as the previously used for the discrete one in Corollary 1 since the coupling signals from the discrete subsystems to the continuous one are cancelled by

the controller (14)-(17) with the compensators and compensating signal fulfilling (20)-(22) while the compensating signal in (22.c) is used to cancel the unsuitable plant zeros. A more general choice of  $r_{1c}(t)$  based on an arbitrary design of  $G_{ic}(D, q)$  ( $i = 3, 4$ ) could be established without difficulty in the same way as addressed in Theorem 1 for the discrete model, although at the expense of more involved calculations.

**Summary of the controller synthesis method and guidelines for particular designs of interest:**

The synthesis of the hybrid controller for the hybrid plant (1) consists of firstly defining the discrete and continuous reference models  $W_{md}(q) = B_{md}(q) / A_{md}(q)$  and  $W_{mc}(q) = B_{mc}(q) / A_{mc}(q)$  for uniformly bounded reference inputs  $r_d[k]$  and  $r_c(t)$ ,  $t \in [kT, (k+1)T)$ . Then,  $u[k]$  and  $u(t)$ ,  $t \in [kT, (k+1)T)$  are generated from (6), with the compensators designed according to Theorem 1 or Corollary 1, and (15)-(18) with the compensators designed according to Theorem 2, respectively. There are several particular designs of practical interest, within the above general framework, which are now described concerning the use of a unique reference model or the way of combining the dynamics of two separate discrete and continuous reference models to improve the performances of the closed-loop system.

**Design 1:** (Continuous-time reference model). The reference input to the continuous-time reference model  $W_{mc}(D)$  is piecewise continuous with discontinuities at sampling instants and being constant inbetween sampling instants and the discrete-time reference model  $W_{md}(q)$  is the z-transform of  $W_{mc}(D)$ . Choose the reference signal as  $r(t) = r_c(t) = r_c[k] = r_d[k] = r[k]$ ,  $t \in [kT, (k+1)T)$ . Thus, the reference output is generated by a unique reference model for all  $t \geq 0$ . The, in general discontinuous, plant input is generated from (6) and Theorem 1, or Corollary 1, for  $t = kT$  and from (15)-(18) and Theorem 2 for  $t \neq kT$ . The main difference of Design 1 with respect to Design 2 below is that the plant input is generated at sampling instants from a discrete-time model - following philosophy while it is generated from a continuous-time model-matching philosophy inbetween sampling instants despite that a unique continuous-time model is available together with

its discretization at sampling instants. In other words, the diophantine equation solving the pole-placement problem at sampling instants is of a discrete nature and it is related to the  $q$ -operator while that used for the continuous dynamics pole-placement is of a continuous nature and it is related to the  $D$ -operator.

**Design 2:** (Continuous-time reference model with the controller using periodic plant reparametrization).  $W_{mc}(D)$  is used as the unique reference model at all time. The use of a discrete-time reference model  $W_{md}(q)$  is omitted in this design. At each new sampling instant  $t = kT$ , the continuous-time description of the plant is reparametrized with the replacement  $P_c(D) Q_d(q)$  with  $P_c(D) Q_d(q) + Q_c(D)N_{cd}^u(D,q)$  in (3), according to Remark 1, since the right-hand-side terms of (1.a) and (1.c) that involve to  $u(t)$  and  $u[k]$  have to be summed up when  $t = kT$ . Thus, (15)-(18) and Theorem 2 are used to generate the control signal for each  $t = kT$  with  $r(t) = r[k] = r_c[k]$ . Subsequently,  $r_{1c}(t) = r_1(t)$  and  $r(t) = r_c(t)$ ,  $t \in (kT, (k+1)T)$  and the plant input  $u(t)$  is generated from (17) and Theorem 2 for  $t \neq kT$ . The main difference of Design 2 with respect to Design 1 is that now the plant input is always generated from Theorem 2 (i.e., from the continuous-time dynamics) with the plant involving a reparametrization at sampling instants (see Remark 1). In other words, the associated pole-placement problem is given by two diophantine equations at and inbetween sampling instants. Those equations are associated to, in general, different plant parametrizations which arise from the fact that the input, state and output signals become additive at sampling instants in the right-hand-sides of eqns. 1.

**Design 3:** (Discrete-time reference model). The plant input is restricted to be piecewise continuous with discontinuities at sampling instants only while being constant inbetween sampling instants, i.e., it is generated by a zero-order-hold and  $u(t) = u[k] = u(kT)$ ,  $t \in (kT, (k+1)T)$ . Thus, only the discrete-time reference model  $W_{md}(q)$  is used in this particular design. Thus,  $r_1[k] = r_{1d}[k]$  and  $r(t) = r[k] = r_d[k]$ . Simple calculus yields  $\dot{r}[k] = \ell u[k]$  with  $\ell = \left( \int_0^T e^{A_c(T-t)} dt \right) b_c$ , which substituted in (2) and (6) yields directly:

$$Q_d y[k] = \bar{P}_d(q) u[k] \tag{23.a}$$

$$u(t) = u[k] = C_{yu}^d(q) y[k] + C_{r_1u}^d(q) r_{1d}[k] \tag{23.b}$$

all  $t \in [kT, (k+1)T)$  with

$$\bar{G}_{1d}(q) = G_{1d}(q) + G_{3d}^T(q) \ell = \bar{G}'_{1d}(q) P_{2d}(q) \tag{23.c}$$

$$\bar{P}_d(q) = P_d(q) + Q_d(q) c^T \ell \tag{23.d}$$

$$C_{yu}^d(q) = \frac{G_{2d}(q)}{L_d(q) - \bar{G}_{1d}(q)} \tag{23.e}$$

$$C_{r_1u}^d(q) = \frac{R_{1d}(q)}{L_d(q) - \bar{G}_{1d}(q)} \tag{23.f}$$

the model-matching problem is solved by applying the controller to the plant by recombining eqns. 26.a while solving the diophantine equation (8) with the replacement  $G_{1d} \rightarrow \bar{G}'_{1d}$  in the solution polynomials  $\bar{G}'_{1d}$  and  $G_{2d}$ , which are unique if  $\deg(\bar{G}'_{1d}) = \deg(P_{1d}) - 1$  and the compensating signal  $r_1(t) = r_1[k] = r_{1d}[k] = R_{1d}^{-1} P_{2d}^{-1} B_{md}^{-1} r[k]$ , all  $t \in [kT, (k+1)T]$  and all nonnegative integer  $k$ .

**Design 4:** (General combined continuous-time and discrete-time reference models with large sampling periods). This design keeps both Objectives 1-2. The discrete-time reference model  $W_{md}(q)$  is designed with a large sampling period compared to the dominant constant of the continuous-time subsystem while keeping Assumption 3. In this context, Objective 2 over the continuous-time reference model  $W_{mc}(D)$  is the basis of the overall design. Objective 1 is used for periodic testing of the closed-loop performance and eventual re-adjustment of the continuous-time model in case of performance's test failure. If such a test fails in terms of excessive deviations of the sampled output from its neighbouring values generated by Objective 1 then either the high-frequency gain of  $W_{mc}(D)$  or its reference input  $r_c(t)$  can be re-updated appropriately. This model re-updating procedure makes justifiable the use of two separate continuous-time and discrete-time reference models and two associated control objectives as stated in the general design procedure. An important advantage is that the possible re-updating could be implemented while keeping two independent continuous-time and discrete-time reference dynamics. Another important key issue which can be extended to all the above designs is that



the continuous-time equations of both the plant and continuous-time reference model can be implemented in practice through a discretization at very low sampling periods compared to the sampling period that regulates the discrete dynamic. All the formulation has been extended to the presence of unmodelled dynamics and noise environments and also it has been tested with numerical examples in both of these situations as well as in the ideal deterministic one, but the obtained results have been omitted by space reasons.

### CONCLUSION

This paper has dealt with the model - following design of a class of single- input single- output linear hybrid systems which consists of a continuous-time subsystem and a digital one which are coupled in general. The digital subsystem is driven by the input samples and it has, in general, couplings with the samples of the continuous substate. The design philosophy is the simultaneous use of a continuous - time model to be matched inbetween sampling instants as well as a discrete- time one which has to be matched at sampling instants. A particular possible design objective, which is included within the general proposed framework, is the choice of the discrete model as the discretization of the continuous - time one to be matched inbetween sampling instants. Another possible particular design is the definition of the discrete model at slow sampling for a periodic on-line testing of performance through testing of either the set point input reference signal or the high frequency gain of the continuous - time reference model with the eventual subsequent re- adjustment of such a model in the case when the above performance' s test is not found to be suitable.

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